Asymptotics and statistics on Fishburn matrices and their generalizations

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ABSTRACT. A direct saddle-point analysis (without relying on any modular forms, identities or functional equations) is developed to establish the asymptotics of Fishburn matrices and a large number of other variants with a similar sum-of-finite-product form for their (formal) general functions. In addition to solving some conjectures, the application of our saddle-point approach to the distributional aspects of statistics on Fishburn matrices is also examined with many new limit theorems characterized, representing the first of their kind for such structures.

CONTENTS

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1. MOTIVATIONS AND BACKGROUND

Fishburn matrices, introduced in the 1970s in the context of interval orders (in order theory) and directed graphs (see [1, 18, 23, 41]), are nonnegative, upper-triangular ones without zero row or column. They have later found to be bijectively equivalent to several other combinatorial structures such as $(\mathbf{2}+\mathbf{2})$-free posets, ascent sequences, pattern-avoiding permutations, pattern-avoiding inversion sequences, Stoimenow matchings, and regular chord diagrams; see, for instance, [6, 14, 21, 30, 35] and Section 2 for more information. In addition to their rich combinatorial connections, the corresponding asymptotic enumeration and the finer distributional properties are equally enriching and challenging, as we will explore in this paper. In particular, while the asymptotics of some classes of Fishburn matrices were known (see, for example, [8, 44]), the stochastic aspects of the major characteristic statistics have remained open up to now.

Zagier, in his influential paper [44] on Vassiliev invariants and quantum modular forms, derived the asymptotic approximation to the number of Fishburn matrices whose entries sum to n

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left(1-(1-z)^j\right) = c\rho^n\, n^{n+1}\left(1+\frac{c_1}{n}+O\big(n^{-2}\big)\right), \tag{1.1}$$

(see OEIS [29] sequence A022493, the Fishburn numbers), where $(c,\rho) = \left(\frac{12\sqrt{6}}{\pi^2}\, e^{\frac{\pi^2}{12}}, \frac{6}{e\pi^2}\right)$ and $c_1 = \frac{11}{24} - \frac{17\pi^2}{144} + \frac{\pi^4}{432}$. Here $[z^n]f(z)$ denotes the coefficient of z^n in the (formal) Taylor expansion of f. For conciseness of notation and readability, *all constant pairs (c,ρ) throughout this paper are generic and may not be the same at each occurrence*; their values will be locally specified.

That the asymptotic approximation (1.1) is remarkable can be viewed in various perspectives. First, the Taylor coefficients of the inner product on the left-hand side of (1.1) alternating in sign, it is unclear if the coefficient of z^n in the sum-of-product expression is positive for all positive n, much less its factorial growth order shown on the right-hand side. Second, since $\frac{6}{\pi^2} < 1$, the right-hand side of (1.1) is exponentially smaller than $n!$, which equals $[z^n]\prod_{1\leqslant j\leqslant n}\big(1-(1-z)^j\big)$. More precisely, we will prove that (see Lemma 12 and 14)

$$\max_{1\leqslant k\leqslant n}\left|[z^n]\prod_{1\leqslant j\leqslant k}\left(1-(1-z)^j\right)\right| = \max_{1\leqslant k\leqslant n}[z^n]\prod_{1\leqslant j\leqslant k}\left((1+z)^j-1\right) = \Theta\big(n^{n+\frac12}\hat\rho^n\big),$$

where $\hat\rho = \frac{12}{e\pi^2} = 2\rho$, so there is indeed a heavy *exponential cancellation* involved in the sum on the left-hand side. Third, in addition to the connection to linearly independent Vassiliev invariants, the Fishburn numbers have now been known to enumerate many different combinatorial objects; see Section 2, OEIS sequence A022493 and [4, 6, 8, 9, 14, 13, 16, 18, 30, 31, 33, 35, 43, 44] for more information. Finally, Zagier's proof of (1.1) relies crucially on the unusual pair of identities

$$\begin{cases} e^{-\frac{z}{24}} \sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \left(1-e^{-jz}\right) = \sum_{n\geqslant 0} \frac{T_n}{n!} \left(\frac{z}{24}\right)^n, \\ \sum_{n\geqslant 0} \frac{T_n}{(2n+1)!} z^{2n+1} = \frac{\sin 2z}{2\cos 3z}, \end{cases} \tag{1.2}$$

where the T_n's are known as Glaisher's T-numbers (see A002439). The asymptotics of T_n is then readily computed by, say using the singularity analysis (see [20]) on the right-hand side of the second identity, which, unlike the formal nature of the first, is analytic in $|z| < \frac{\pi}{6}$. These identities are a consequence of partial summation, Euler's pentagonal number theorem, functional equations, Dirichlet series and Mellin transform techniques; see [44, 45]. What appears to be more important in subsequent developments is that T_n is essentially the value defined by the analytic continuation of some Dirichlet series at $-2n-1$, and the study of the identities (1.2) is thus closely connected to algebraic and analytic number theory, in addition to their hypergeometric q-series nature and resurgence aspect [11].

Many similar pairs of relations such as (1.2) are now known; see also [5, 37]. For example, the tangent numbers (A000182) satisfy the pair of generating functions

$$\begin{cases} \sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \tanh(2jz) = \sum_{n\geqslant 0} \frac{a_n}{n!} z^n, \\ \sum_{n\geqslant 0} \frac{a_n}{(2n+1)!} z^{2n+1} = \tan z, \end{cases} \tag{1.3}$$

which is a special case of general theorems in [5]. It follows that

$$[z^n] \sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \tanh(2jz) \simeq c\rho^n n^{n+1}, \quad \text{with} \quad (c,\rho) = \left(\tfrac{16\sqrt{2}}{\pi^2}, \tfrac{16}{e\pi^2}\right),$$

which has the same asymptotic pattern as (1.1)—a universal aspect we will exhibit through several analytic schemes and many examples in this paper. *Here and throughout this paper, the asymptotic relation*

$$a_n = b_n\left(1+O\left(n^{-1}\right)\right) \quad \text{is abbreviated as} \quad a_n \simeq b_n. \tag{1.4}$$

Many other pairs similar in spirit to (1.2) and (1.3) can be found in [5, 25, 37].

Along another direction, Bringmann *et al.* [8] extended Zagier's proof strategy and derived the asymptotic approximation to the number of primitive row-Fishburn matrices with entries summing to n

$$[z^n] \sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \left((1+z)^j - 1\right) \simeq c\rho^n n^{n+\frac{1}{2}}, \text{ with } (c,\rho) = \left(\tfrac{12}{\pi^{3/2}} e^{-\frac{\pi^2}{24}}, \tfrac{12}{e\pi^2}\right). \tag{1.5}$$

[A primitive row-Fishburn matrix is a binary upper-triangular one without zero rows.] Their approach to deriving (1.5) relies on various properties of the function ($\sigma(q)$ in [8])

$$R(q) := \sum_{k\geqslant 0} \frac{q^{\frac{1}{2}k(k+1)}}{(1+q)\cdots(1+q^k)} = 1 + \sum_{k\geqslant 0} (-1)^k q^{k+1} \prod_{1\leqslant j\leqslant k} \left(1-q^j\right), \tag{1.6}$$

first appeared in Ramanujan's lost notebook, with many unusual properties discovered since Andrews's paper [2]; see [3, 10] and A003406 for more information. Very roughly, since

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big((1+z)^j-1\big) = \frac{(-1)^n}{2}[z^n]R(1-z),$$

and $e^{-z} = 1-z+O(|z|^2)$ for small $|z|$, the approach begins by working out the asymptotics of $[z^n]R\big(e^{-z}\big)$. The bridge between $[z^n]R(1-z)$ and $R\big(e^{-z}\big)$ can then be linked through a direct change of variables and straightforward arguments because z is very close to zero (the arguments used in [44] and [8] relying instead on the asymptotics of the Stirling numbers of the first kind); see Section 4.2 for more details.

The asymptotics of $[z^n]R\big(e^{-z}\big)$ is derived by first defining the Dirichlet series

$$D(s) := \sum_{n\geqslant 1} n^{-s}[q^{n-1}]R\big(q^{24}\big),$$

which can be meromorphically continued into the whole plane. Since, by standard Mellin transform techniques (see, e.g., [19]),

$$[z^n]e^{-z}R\big(e^{-24z}\big) = \frac{(-1)^n}{n!}\,D(-n),$$

the crucial asymptotics of $D(-n)$ needed is then derived by the functional equation satisfied by certain function defined on $D(s)$ (similar to that satisfied by Riemann's zeta function); see [8, 10] and Section 4 for more details.

Our aim in this paper is to develop a *direct, self-contained* approach to deriving (1.1) and (1.5) in a systematic way without relying on any functional equations (satisfied by Dirichlet series) or identities such as (1.2) and (1.3), which are not available in more general contexts with a similar sum-of-finite-product form for the generating functions. Our approach is based instead on a fine, double saddle-point analysis and, although less deep in nature, has the additional advantages of being applicable to a large number of problems whose (formal) generating functions assume a similar form. In particular, we can derive the asymptotics of generalized Fishburn matrices whose entries are restricted to lie in any multiset of nonnegative integers containing particularly 0. The approach is also applied to more than two dozens of OEIS sequences and to confirm a conjecture of Jelínek in [30] concerning self-dual Fishburn matrices. Furthermore, we will also address the corresponding distributional aspect by extending the same saddle-point approach and derive the limit laws of some typical statistics in wide generality, which answers particularly an open problem raised by Bringmann *et al.* [8] and Jelínek [30].

Our approach is best illustrated through the prototypical (rational) sequence

$$a_n := [z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big(e^{jz}-1\big) = \frac{(-1)^n}{2}[z^n]R\big(e^{-z}\big), \tag{1.7}$$

where R is defined in (1.6), for which we will show *inter alia* that

$$a_n = c\rho^n n^{n+\frac12}\Big(1+\frac{\nu_1}{n}+\frac{\nu_2}{n^2}+O\big(n^{-3}\big)\Big), \text{ with } (c,\rho)=\big(\tfrac{12}{\pi^{3/2}},\tfrac{12}{e\pi^2}\big), \tag{1.8}$$

where $\nu_1 = \frac{24-\pi^2}{288}$ and $\nu_2 = \frac12\nu_1^2$; see (4.1) for an asymptotic expansion. Here the integer sequence $\{a_n n!\}$ corresponds to A158690 in the OEIS. *Throughout this paper, we do not distinguish between* *ordinary and exponential generating functions, and focus only on the large-n asymptotics of the coefficients, so whether the sequence is integer or not is immaterial for our purposes.*

Once the asymptotics (1.8) is available, we extend our approach to the sequences of the form

$$[z^n]\sum_{k\geqslant 0} d(z)^{k+\omega_0} \prod_{1\leqslant j\leqslant k} \left(e(z)^{j+\omega}-1\right)^{\alpha}, \tag{1.9}$$

for $\alpha \in \mathbb{Z}^+$ and $\omega_0, \omega \in \mathbb{C}$. Here the generating functions $d(z)$ and $e(z)$ satisfy $d(0) > 0, e(0) = 1$ and $e'(0) \neq 0$.

Our result (Theorem 21) for the general form (1.9) will not only be applied to derive the large-n asymptotics of many sequences in the literature and the OEIS (see Section 6), but also be sufficient to re-derive (1.1) because of the identity (in the sense of formal power series) due to Andrews and Jelínek [4]

$$\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left(1-(1-z)^j\right) = \sum_{k\geqslant 0}(1-z)^{-k-1}\prod_{1\leqslant j\leqslant k}\left((1-z)^{-j}-1\right)^2.$$

This and other examples of similar types are collected in Section 6.

In addition to its usefulness in univariate asymptotics, our formulation (1.9) is also effective in dealing with (bivariate asymptotics) the limiting distributions of various statistics on random Fishburn matrices with or without restriction on their entries. More precisely, we consider upper-triangular matrices whose entries belong to Λ, a multiset of nonnegative integers with the generating function $\Lambda(z) = 1 + \lambda_1 z + \lambda_2 z^2 + \cdots$. We then define two classes of matrices: (i) Λ-*row-Fishburn* ones without zero row, and (ii) Λ-*Fishburn* ones without zero row or zero column. The statistics examined and their limit laws are summarized in Table 1, where we assume a uniform distribution on the set of all possible such matrices with the same entry-sum.

$\lambda_1 > 0$	Random Λ-row-Fishburn matrices	Random Λ-Fishburn matrices
First row sum	Zero-Tuncated-Poisson$(\log 2)$	Normal$(\log n, \log n)$
Diagonal sum	Normal$(\log n, \log n)$	Normal$(2\log n, 2\log n)$
$\frac{1}{2}\big(n-\#(1\text{s})\big)$	$\begin{cases}\text{Poisson}\left(\frac{\lambda_2\pi^2}{12\lambda_1^2}\right), & \text{if } \lambda_2 > 0\\ \text{degenerate}, & \text{if } \lambda_2 = 0\end{cases}$	$\begin{cases}\text{Poisson}\left(\frac{\lambda_2\pi^2}{6\lambda_1^2}\right), & \text{if } \lambda_2 > 0\\ \text{degenerate}, & \text{if } \lambda_2 = 0\end{cases}$

Table 1. The various asymptotic distributions of the three statistics in large random Λ-row-Fishburn and Λ-Fishburn matrices with entries belonging to a given multiset of nonnegative integers Λ (containing 0 *exactly once and* 1 *at least once). Here n is the sum of all entries in the matrix.*

In particular, when Λ is the set of nonnegative integers, the first row-size in random Fishburn matrices also arises in many different contexts under different guises; see Section 2.3 for more information. Our limit results thus have many different interpretations and implications.

The proof of these limit laws requires the full power of our setting (1.9) where some parameters or coefficients are themselves complex variables, as well as the Quasi-Powers Framework (see [20, 26, 27]), which is a simple synthetic scheme for deriving asymptotic normality and some of its quantitative refinements.

From Table 1 we see that in a typical random Fishburn matrix (when all matrices of the same entry-sum are equally likely), entries equal to 1 are almost everywhere, those to 2 appear like a Poisson distribution, and the rest is asymptotically negligible. In other words, a random Λ-Fishburn matrix is asymptotically close to a random primitive Λ-Fishburn matrix in which only 0 and 1 are allowed as entries. This is also intuitively connected to the asymptotic logarithmicality of the first row sum and the diagonal sum in random Fishburn matrices.

If we regard a Fishburn matrix of size n as an integer partition of n arranged on upper-triangular square matrices without zero row or column, then it is of interest to compare the number of 1s in both models (assuming a uniform distribution in both cases). While the number of 1s in a random integer partition of n follows asymptotically an exponential distribution with mean of order $\sqrt{n}$, the number of 1s in a random Fishburn matrix of size n is asymptotically n minus a Poisson distribution, which means much less random and more deterministic (variance being bounded), making such a random model probably less useful for general stochastic modeling purposes.

What is less expected is that if the smallest nonzero entry is 2 (namely, $1 \notin \Lambda$), then its occurrence in the resulting random matrices becomes more interesting: not only is there a change of limit law according as $\lambda_3 > 0$ or $\lambda_3 = 0$ from normal to Poisson, but more variability is gained when $\lambda_3 > 0$; see Table 2 for a summary of results.

More precisely, we extend further our study in Section 8 to the situation when $\lambda_1 = 0$ but $\lambda_2 > 0$ in Λ-Fishburn matrices, which turns out to have a very similar analytic context to the self-dual (or persymmetric) Fishburn matrices when $\lambda_1 > 0$, which was considered in [30] in the cases when $\Lambda = \{0, 1\}$ and $\Lambda = \mathbb{Z}_{\geqslant 0}$. We adopt the same framework (1.9) and address the asymptotics when $\lambda_1 = 0$ and $\lambda_2 > 0$. Such a formulation is, as in the case of $\lambda_1 > 0$, not only useful for the asymptotic enumeration of matrices of large size, but also effective in characterizing the finer stochastic behaviors of the random matrices, whether they are Fishburn with $\lambda_1 = 0$ and $\lambda_2 > 0$ or self-dual Fishburn with $\lambda_1 > 0$.

While the logarithmic behaviors in the first row sum and the diagonal sum are similar as in Table 1, the limit laws of the occurrences of the smallest nonzero entries seem less predicted, notably in the case when 2 is the smallest nonzero entry. Roughly, the periodicity resulted from the omnipresent factor 2 in the class of Λ-Fishburn matrices without using 1 as entries does change drastically the behavior from being bounded Poisson to normal with mean and variance both asymptotic to $\tau\sqrt{n}$ (or indeed a Poisson distribution with unbounded mean $\tau\sqrt{n}$; see Section 8.2).

Our formulation and results include as a special case the following asymptotic approximation to self-dual Fishburn numbers

$$[z^n]\sum_{k\geqslant 0}(1-z)^{-k-1}\prod_{1\leqslant j\leqslant k}\big((1-z^2)^{-j}-1\big) = ce^{\beta\sqrt{n}}\rho^{\frac12 n}n^{\frac12(n+1)}\big(1+O\big(n^{-\frac12}\big)\big), \tag{1.10}$$

confirming Jelínek's conjecture in [30], where

$$(c,\rho) = \Big(\frac{6}{\pi^{3/2}}\,e^{\frac{\pi^2}{24}-\frac14+\frac{3}{2\pi^2}(\log 2)^2}, \frac{6}{e\pi^2}\Big)$$

and $\beta = \frac{\sqrt{6}\log 2}{\pi}$. The constant $c \approx 1.36195\,1039$ is given in [30] only in approximate numerical form. Note specially the change of the dominant exponential part $\rho^{\frac12 n}n^{\frac12(n+1)}$, and the presence of the extra factor $e^{\beta\sqrt{n}}$ when compared to (1.1) and (1.5). It will become clear through our general frameworks in Sections 5 and 8 why the asymptotic patterns are different although the generating functions look similar.

	Random Λ-Fishburn matrices with $\lambda_1 = \cdots = \lambda_{2m-1} = 0$ and $\lambda_2, \lambda_{2m+1} > 0$	Random self-dual Λ-Fishburn matrices with 1s ($\lambda_1 > 0$)
First row sum	$\mathrm{Normal}(\log n, \log n)$	$\mathrm{Normal}(\log n, \log n)$
Diagonal sum	$\mathrm{Normal}(2\log n, 2\log n)$	$2 \cdot \mathrm{Normal}(\log n, \log n)$
# smallest nonzero entries	$\begin{cases} \frac{1}{2}n - \frac{3}{2} \cdot \mathrm{Normal}(\tau\sqrt{n}, \tau\sqrt{n}), \\ \quad \text{if } m = 1 \\ \frac{1}{2}n - 2 \cdot \mathrm{Poisson}(\frac{\lambda_4\pi^2}{6\lambda_2^2}), \\ \quad \text{if } m \geqslant 2, \lambda_4 > 0, n \text{ even} \\ \frac{1}{2}(n-2m-1) - 2 \cdot \mathrm{Poisson}(\frac{\lambda_4\pi^2}{6\lambda_2^2}), \\ \quad \text{if } m \geqslant 2, \lambda_4 > 0, n \text{ odd} \\ \text{degenerate, if } \lambda_3 = \lambda_4 = 0 \end{cases}$	$\begin{cases} n - 2 \cdot \mathrm{Poisson}\big(\frac{\lambda_2}{\lambda_1}\log 2\big) \\ \quad * \, 4 \cdot \mathrm{Poisson}\big(\frac{\lambda_2\pi^2}{12\lambda_1^2}\big), \\ \quad \text{if } \lambda_2 > 0 \\ \text{degenerate, if } \lambda_2 = 0 \end{cases}$

Table 2. The asymptotic distributions of the three statistics in large random Λ-Fishburn and self-dual Λ-Fishburn matrices with entries belonging to a given multiset of nonnegative integers Λ (containing 0 exactly once and with or without 1s). Here n is the sum of all entries in the matrix, and $\tau := \frac{\lambda_3\pi}{2\sqrt{3}\lambda_2^{3/2}}$.

Similarly, for primitive self-dual Fishburn numbers, we also have

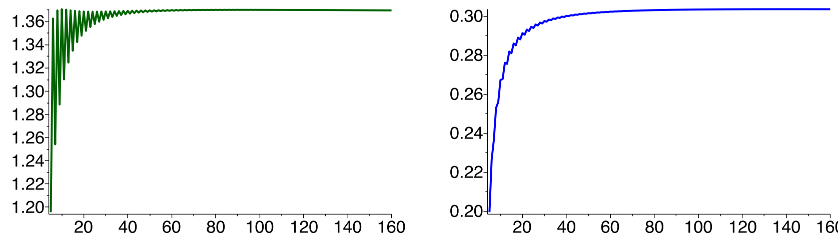
$$[z^n]\sum_{k\geqslant 0}(1+z)^{k+1}\prod_{1\leqslant j\leqslant k}\big((1+z^2)^j - 1\big) = ce^{\beta\sqrt{n}}\rho^{\frac{1}{2}n}n^{\frac{1}{2}(n+1)}\big(1 + O\big(n^{-\frac{1}{2}}\big)\big), \tag{1.11}$$

where $(c, \rho) = \big(\frac{3}{\pi^{3/2}}\,e^{-\frac{\pi^2}{24}-\frac{1}{4}+\frac{3}{2\pi^2}(\log 2)^2}, \frac{6}{e\pi^2}\big)$ and $\beta = \frac{\sqrt{6}\log 2}{\pi}$. We see that the two asymptotic approximations (1.10) and (1.11) differ by a factor of $2e^{\frac{\pi^2}{12}} \approx 4.55$, or, about 21.9% of self-dual Fishburn matrices are primitive; see Figure 1.1. Of course, by comparing these estimates with (1.1), we see that the proportion of self-dual Fishburn matrices is asymptotically negligible (indeed factorially small).

This paper is structured as follows. In the next section, we outline the background on the Fishburn matrices, and then derive the generating functions that will be analyzed in later sections. Then we describe the saddle-point method in detail in Section 3 which will then be used and modified throughout this paper, with the finer asymptotic expansions briefly discussed in Section 4. Then we consider the general framework (1.9) in Section 5 by extending the saddle-point analysis of Section 3. Asymptotics of generalized Fishburn matrices as well as other univariate examples are collected and discussed in Section 6. The asymptotic distributions of the statistics on random Fishburn matrices as those given in Table 1 are then derived in Section 7. The extension of (1.9) to the case when $e_1 = 0$ and $e_2 > 0$ is examined in Section 8, together with univariate and bivariate applications.

Notations. As mentioned at the beginning of this section, (c, ρ) is used generically and will always be locally defined. Other generic and mostly local symbols include c_0, $c(\cdot)$, ε, f, and a_n; their values will be specified whenever ambiguities may occur.

$$\frac{\text{LHS of (1.10)}}{e^{\beta\sqrt{n}}\rho^{\frac{1}{2}n}n^{\frac{1}{2}(n+1)}}\left(1-\frac{0.2}{\sqrt{n}}\right) \qquad \frac{\text{LHS of (1.11)}}{e^{\beta\sqrt{n}}\rho^{\frac{1}{2}n}n^{\frac{1}{2}(n+1)}}\left(1-\frac{1.5}{\sqrt{n}}\right)$$



$$\frac{6}{\pi^{3/2}}\,e^{\frac{\pi^2}{24}-\frac{1}{4}+\frac{3}{2\pi^2}(\log 2)^2} \approx 1.362 \qquad \frac{3}{\pi^{3/2}}\,e^{-\frac{\pi^2}{24}-\frac{1}{4}+\frac{3}{2\pi^2}(\log 2)^2} \approx 0.299$$

Figure 1.1. Numerical convergence of the two ratios $\frac{\text{LHS of (1.10)}}{e^{\beta\sqrt{n}}\rho^{\frac{1}{2}n}n^{\frac{1}{2}(n+1)}}$ and $\frac{\text{LHS of (1.11)}}{e^{\beta\sqrt{n}}\rho^{\frac{1}{2}n}n^{\frac{1}{2}(n+1)}}$ (with proper corrections for the O-terms) to their respective limit c.

2. Fishburn matrices and related combinatorial objects

Throughout this paper, the *size* of a matrix is defined to be the sum of all its entries. Similarly, we write the size of a row or a column or the diagonal to represent their respective sum.

Definition 1 (Fishburn matrix)**.** A Fishburn matrix is an upper-triangular square one with nonnegative integer entries such that no row or no column consists solely of zeros.

As a succinct representation tool for interval orders (see [23, 18]), Fishburn matrices (called IO-matrices in [23], characteristic matrices in [17, 18], and composition matrices in [12], and named so in [9]) offer not only algorithmic but also combinatorial advantages, and over the years their study was largely enriched by the corresponding developments in combinatorial enumeration and bijections, following notably Bousquet-Mélou et al.'s paper [6]. In particular, the useful database OEIS [29] played a key role in linking the various structures in different areas some of which will be briefly described later.

Closer to our interest here, the enumeration of Fishburn matrices of a given *dimension* was already investigated in the early papers [1, 23], and recursive algorithms were later proposed for computing matrices of a given size (see e.g.,[24, 39]), culminating in the definitive work of Zagier [44], where, through the proper use of generating functions, effective asymptotic approximations (1.1) for Fishburn matrices of large size are derived.

We describe Fishburn matrices in this section, together with some of their variants and generalizations. We also derive the bivariate generating functions for some statistics that will be examined in more detail in later sections.

2.1. **Fishburn matrices and their variants.** Recall that the Fishburn numbers (A022493) enumerate Fishburn matrices of a given size and can be computed by the generating function

$$\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left(1-(1-z)^j\right) = 1 + z + 2z^2 + 5z^3 + 15z^4 + 53z^5 + 217z^6 + \cdots .$$

For example, all 15 Fishburn matrices of size 4 are depicted in Figure 2.1.

$$(4)\ \begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}2&1\\0&1\end{pmatrix}\begin{pmatrix}1&1\\0&2\end{pmatrix}\begin{pmatrix}2&0\\0&2\end{pmatrix}\begin{pmatrix}3&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&3\end{pmatrix}$$

$$\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}2&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&2&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}\begin{pmatrix}1&1&0\\0&0&1\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

Figure 2.1. All 15 Fishburn matrices of size 4. The occurrence of 1 is seen to be predominant.

From a combinatorial viewpoint, the Fishburn numbers also enumerate several seemingly unrelated structures some of which are listed as follows; see [6, 9, 14, 13, 16, 18, 21, 30, 33, 35, 43] for the bijective and algebraic proofs of these equinumerosity.

- *Ascent sequences of length* n, which are sequences of nonnegative integers $(x_1, x_2, \dots, x_n)$ such that for each i, $0 \leqslant x_i \leqslant 1 + |\{j : 1 \leqslant j \leqslant i-2 \text{ and } x_j < x_{j+1}\}|$.
- $(\mathbf{2-1})$*-avoiding inversion sequences of length* n: these are sequences $x = (x_1, x_2, \dots, x_n)$ such that $0 \leqslant x_i < i$ and there exists *no* $i < j$ such that $x_i = x_j + 1$.
- $(2|3\bar{1})$*-avoiding permutations of* n *elements*, which are permutations π without subsequence $\pi_i\pi_{i+1}\pi_j$ such that $\pi_i - 1 = \pi_j$ and $\pi_i < \pi_{i+1}$.
- $(\mathbf{2+2})$*-free posets of* n *elements*: these are posets $(P, \prec)$ with *interval representations*, namely, for each $x \in P$, a real closed interval $[\ell_x, r_x]$ is associated to x such that $x \prec y$ in P exactly when $r_x < \ell_y$.
- *Stoimenow matchings of length* $2n$: A matching of the set $[2n] = \{1, 2, \dots, 2n\}$ is a partition of $[2n]$ into subsets (called *arcs*) of size exactly two. A Stoimenow matching is one without nested pair of arcs such that either the openers or the closers are next to each other.
- *Regular linearized chord diagrams of length* $2n$: A regular linear chord diagram is a fixed-point free involution τ on the set $[2n]$ such that if $[i, i+1] \subset [\tau(i+1), \tau(i)]$ whenever $\tau(i+1) < \tau(i)$.

Two variants of Fishburn matrices, *row-Fishburn matrices* and *self-dual Fishburn matrices* were introduced by Jelínek [30] during his study on refined enumeration of self-dual interval orders. *Row-Fishburn matrices* are upper-triangular ones with non-negative integer entries such that no row is composed solely of zeros. The corresponding generating function satisfies

$$\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big((1-z)^{-j}-1\big) = 1 + z + 3z^2 + 12z^3 + 61z^4 + 380z^5 + 2815z^6 + \cdots, \tag{2.1}$$

where the coefficient of z^n equals the number of row-Fishburn matrices of size n.

Furthermore, a matrix is *primitive* if all entries are either 0 or 1. Substituting z by $\frac{z}{1+z}$ leads to the generating function for the *primitive row-Fishburn number* (A179525)

$$\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big((1+z)^{j}-1\big) = 1 + z + 2\,z^2 + 7\,z^3 + 33\,z^4 + 197\,z^5 + 1419\,z^6 + \cdots. \tag{2.2}$$

Reversely, (2.1) can be obtained from (2.2) by substituting z with $\frac{z}{1-z}$.

On the other hand, a Fishburn matrix is *self-dual* if it is persymmetric, or symmetric with respect to the northeast-southwest diagonal. The number of self-dual Fishburn matrices of size n satisfies (1.10), and the number of *primitive self-dual Fishburn number* of size n is given in (1.11). In particular, for the number of self-dual Fishburn matrices of a give size, we have

$$\sum_{k\geqslant 0}(1-z)^{-k-1}\prod_{1\leqslant j\leqslant k}\big((1-z^2)^{-j}-1\big) = 1+z+2\,z^2+3\,z^3+7\,z^4+13\,z^5+33\,z^6+\cdots,$$

so that 7 among the 15 Fishburn matrices of size 4 are self-dual, as can be readily checked with Figure 2.1.

2.2. **Generalized Fishburn matrices.** We now extend the matrices by allowing more flexible entries. Let Λ be a multiset of nonnegative integers with the generating function

$$\Lambda(z) := 1+\sum_{\lambda\in\Lambda} z^{\lambda} = 1+\lambda_1 z+\lambda_2 z^2+\cdots. \tag{2.3}$$

Assume *throughout this paper except in Sections 8.1, 8.3 and 8.4 that* $\lambda_1 > 0$, so that $\{0,1\}\in\Lambda$.

Definition 2 (Λ-Fishburn matrix)**.** An upper-triangular matrix is called Λ-*Fishburn* if every row and column has non-zero size, and all entries lie in the set Λ.

Definition 3 (Λ-row-Fishburn matrix)**.** An upper-triangular matrix is called Λ-*row-Fishburn* if all entries lie in the set Λ without zero row.

Proposition 1. *The number of Λ-row-Fishburn matrices of size n is given by*

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big(\Lambda(z)^j-1\big), \tag{2.4}$$

and that of Λ-Fishburn matrices by

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big(1-\Lambda(z)^{-j}\big) = [z^n]\sum_{k\geqslant 0}\Lambda(z)^{k+1}\prod_{1\leqslant j\leqslant k}\big(\Lambda(z)^j-1\big)^2. \tag{2.5}$$

Proof. The first generating function (2.4) follows from the definition of Λ-row-Fishburn matrices. For a Λ-row-Fishburn matrix of dimension k, and for any j, $1\leqslant j\leqslant k$, the generating function of the $(k-j+1)$-st row counted by the size (variable z) is $\Lambda(z)^j-1$. As a result, the generating function for Λ-row-Fishburn matrices of dimension k is given by $\prod_{1\leqslant j\leqslant k}(\Lambda(z)^j-1)$. Summing over all k leads to (2.4).

On the other hand, the generating function for primitive Fishburn matrices is given by (including the constant 1 for the empty matrix; see [30])

$$\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big(1-(1+z)^{-j}\big).$$

Substituting $1+z$ by $\Lambda(z)$ yields the generating function for Λ-Fishburn matrices in the general case.

The right-hand side of the identity (2.5) follows from the following q-identity due to Andrews and Jelínek [4]

$$\begin{aligned}\sum_{k\geqslant 0}u^k\prod_{1\leqslant j\leqslant k}&\left(1-\frac{1}{(1-s)(1-t)^{j-1}}\right)\\ &=\sum_{k\geqslant 0}(1-s)(1-t)^k\prod_{1\leqslant j\leqslant k}\Big(\big(1-(1-s)(1-t)^{j-1}\big)\big(1-u(1-t)^j\big)\Big),\end{aligned} \tag{2.6}$$

by substituting $u=1$ and $s=t=1-\Lambda(z)$ on both sides. $\square$

2.3. **Statistics on Fishburn matrices.** The study of statistics on the Fishburn structures traces back to the work by Andresen and Kjeldsen [1] in the context of transitively directed graphs (see also [30]), where they studied the numbers of primitive Fishburn matrices counted by the dimension and by the size of the first row (with the notation $\xi(n,k)$ in [1]).

Stoimenow [39] found a recursive formula for the numbers of regular linearized chord diagram with a given length of the leftmost chord. Subsequently, it was discovered [6, 21, 30, 34, 35, 43] that these numbers are equivalent to the following ones:

- the sum of entries in the first row (or the last column) of Fishburn matrices of size n;
- the number of minimal (or maximal) elements in $(2+2)$-free posets of size n;
- the maximal entries (or right-to-left minimal entries, or the number of zeros) in ascent sequences of length n;
- the maximal entries in $(\mathbf{2}-\mathbf{1})$-avoiding inversion sequences of length n;
- the length of the initial run of openers in Stoimenow matchings of length $[2n]$;
- the length of the initial decreasing run in $(2|3\bar{1})$-avoiding permutations of length n;
- the number of left-to-right minima (or left-to-right maxima; or right-to-left maxima) in $(2|3\bar{1})$-avoiding permutations of length n.

These statistics are classified as Stirling statistics (or roughly statistics with logarithmic mean and variance); see [21]. On the other hand, the diagonal size that we address in this paper represents another Stirling statistic, which may also have interpretations in terms of other structures.

Parallel to the Stirling statistics, the Eulerian statistics (or roughly statistics with linear mean and variance) have also been intensively studied but the corresponding (asymptotic) distributional aspect has remained open and will be addressed elsewhere.

2.4. **Bivariate generating functions for generalized Fishburn matrices.** We study the asymptotic distributions of the following three random variables on random Λ-Fishburn and Λ-row-Fishburn matrices, assuming a uniform distribution on the set of all size-n matrices: the size of the first row, the size of the diagonal, and the number of occurrences of 1.

The approach we use to characterize the corresponding limit laws relies heavily on the corresponding bivariate generating functions and our double saddle-point analysis. We derive the required generating functions in this subsection. We use the convention that $f(z,v)$ is the bivariate generating function for the quantity X if $[z^n v^m]f(z,v)$ denotes the number of Λ-row-Fishburn matrices of size n with $X=m$.

Proposition 2 (Statistics on Λ-row-Fishburn matrices)**.** *We have the following bivariate generating functions with z marking the matrix size.*

(i) The size of the first row

$$1+\sum_{k\geqslant 0}\big(\Lambda(vz)^{k+1}-1\big)\prod_{1\leqslant j\leqslant k}\big(\Lambda(z)^j-1\big), \tag{2.7}$$

(ii) the size of the diagonal

$$\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big(\Lambda(vz)\Lambda(z)^{j-1}-1\big), \tag{2.8}$$

and

(iii) the number of 1*s*

$$\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big((\Lambda(z)+\lambda_1(v-1)z)^j-1\big). \tag{2.9}$$

Proof. Given a Λ-row-Fishburn matrix of dimension $k+1$, the generating function for the first row size (marked by vz) is $\Lambda(vz)^{k+1}-1$, the remaining k rows contributing $\prod_{1\leqslant j\leqslant k}\big(\Lambda(z)^j-1\big)$, as in the proof of (2.4). The proofs for the other two parameters are similar and thus omitted. □

For Λ-Fishburn matrices, the proof is less straightforward and we need a fine-tuned version of Jelínek's Theorem 2.1 in [30] in order to enumerate both the first row and the diagonal sizes.

Let $\mathscr{P}$ denote the set of primitive (binary) Λ-Fishburn matrices. Define first

$$\begin{aligned}&G_k(t,u,v,w,x,y)\\&\quad:=\sum_{(M_{i,j})_{k\times k}\in\mathscr{P}} t^{M_{k,k}}u^{\sum_{2\leqslant j<k}M_{j,k}}v^{\sum_{2\leqslant j<k}M_{j,j}}w^{\sum_{1<i<j<k}M_{i,j}}x^{M_{1,k}}y^{\sum_{1\leqslant j<k}M_{1,j}},\end{aligned}$$

so that t marks the lower-right corner (which is always 1), u the size of the last column except the two ends, v the size of the diagonal except the two ends, w the size of all interior cells, x the upper-right corner, and y the size of the first row except the upper-right corner.

Lemma 3. *The generating function* $F(s,t,u,v,w,x,y):=\sum_{k\geqslant 2}G_k(t,u,v,w,x,y)s^k$ *satisfies*

$$\begin{aligned}&F(s,t,u,v,w,x,y)\\&\quad=t\sum_{k\geqslant 0}\frac{s^{k+2}y(1+x)(1+y)^k}{(1+u)(1+v)(1+w)^k-1}\prod_{0\leqslant j\leqslant k}\frac{(1+u)(1+v)(1+w)^j-1}{1+s\big((1+u)(1+w)^j-1\big)}.\end{aligned}\tag{2.10}$$

Proof. (Sketch) By definition, it is clear that $G_1=x$ and $G_k(t,u,v,w,x,y)=tG_k(1,u,v,w,x,y)$ for $k\geqslant 2$. By the recursive construction used in [30, Lemma 2.8], we derive the recurrence relation

$$\begin{aligned}&G_{k+1}(1,u,v,w,x,y)\\&\quad=G_k(u+v+uv,u,v+w+vw,w,x+y+xy,y)-vG_k(1,u,v,w,x,y)\\&\quad=(u+v+uv)G_k(1,u,v+w+vw,w,x+y+xy,y)-vG_k(1,u,v,w,x,y),\end{aligned}$$

for $k\geqslant 2$. Then from this and the iterative arguments used in [30], we deduce (2.10); see [30] for details. □

Proposition 4 (Statistics on Λ-Fishburn matrices)**.** *We have the following bivariate generating functions with z marking the matrix size.*

(i) The size of the first row

$$\begin{aligned}&\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big(1-\Lambda(vz)^{-1}\Lambda(z)^{1-j}\big)\\&\quad=\Lambda(vz)\sum_{k\geqslant 0}\Lambda(z)^k\prod_{1\leqslant j\leqslant k}\big((\Lambda(vz)\Lambda(z)^{j-1}-1)(\Lambda(z)^j-1)\big),\end{aligned}\tag{2.11}$$

(ii) the size of the diagonal

$$
\begin{aligned}
&1+\Lambda(vz)+(\Lambda(vz)-1)^2\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big(\Lambda(vz)-\Lambda(z)^{-j}\big)\\
&\qquad=\Lambda(vz)\sum_{k\geqslant 0}\Lambda(z)^k\prod_{1\leqslant j\leqslant k}\big(\Lambda(vz)\Lambda(z)^{j-1}-1\big)^2,
\end{aligned}
\tag{2.12}
$$

and

(iii) the number of 1*s*

$$
\begin{aligned}
&\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big(1-(\Lambda(z)+\lambda_1(v-1)z)^{-j}\big)\\
&\qquad=\sum_{k\geqslant 0}(\Lambda(z)+\lambda_1(v-1)z)^{k+1}\prod_{1\leqslant j\leqslant k}\big((\Lambda(z)+\lambda_1(v-1)z)^j-1\big)^2.
\end{aligned}
\tag{2.13}
$$

Proof. (i) For the size of the first row, we have, by (2.10), that the generating function for primitive Fishburn matrices with size marked by z and the first row size by v is given by

$$
\begin{aligned}
F(1,z,z,z,z,vz,vz)&=\sum_{k\geqslant 0}\frac{vz^2(1+vz)^{k+1}(1+z)^{k+1}}{(1+z)^{k+2}-1}\prod_{0\leqslant j\leqslant k}\big(1-(1+z)^{-j-2}\big),\\
&=vz\sum_{k\geqslant 1}(1+vz)^k\prod_{1\leqslant j\leqslant k}\big(1-(1+z)^{-j}\big).
\end{aligned}
$$

Then, by the Andrew-Jelínek identity (2.6) and the identity [4, Eq. (1)], we obtain

$$
\begin{aligned}
&1+vz+F(1,z,z,z,z,vz,vz)\\
&\qquad=(1+vz)\sum_{k\geqslant 0}(1+z)^k\prod_{1\leqslant j\leqslant k}\big((1+vz)(1+z)^{j-1}-1\big)\big((1+z)^j-1\big).
\end{aligned}
$$

Substituting $1+vz$ by $\Lambda(vz)$ and $1+z$ by $\Lambda(z)$ proves (2.11).

Alternatively, it is known that the generating function for the size of the first row (marked by v) is given by (see [21, 33])

$$
\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}(1-(1+vz)^{-1}(1+z)^{1-j}).
$$

Substituting $1+vz$ by $\Lambda(vz)$ and $1+z$ by $\Lambda(z)$ gives the alternative generating function for the first row size

$$
\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big(1-\Lambda(vz)^{-1}\Lambda(z)^{1-j}\big),
$$

which, equals, by substituting $u=1$, $s=1-\Lambda(vz)$ and $t=1-\Lambda(z)$ in (2.6), the same generating function on the right-hand side of (2.11).

(ii) For the size of the diagonal, we have, again, by (2.10), the generating function

$$
1+vz+F(1,v^2z,z,vz,z,z,z)=1+vz+(vz)^2\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big(1+vz-(1+z)^{-j}\big).
$$

The same substitutions $1+vz \mapsto \Lambda(vz)$ and $1+z \mapsto \Lambda(z)$ give the left-hand side of (2.12). Applying now (2.6) with $u = 1+vz$, $s = 1-(1+vz)(1+z)$ and $t = -z$, and then using the same substitutions, we obtain the right-hand side of (2.12).

(iii) The generating functions (2.13) for the number of 1s follow from substituting $\Lambda(z)$ by $\Lambda(z) + \lambda_1(v-1)z$ in (2.5).

□

3. Asymptotics of the prototype sequence A158690

Consider the sequence $a_n := [z^n]A(z)$, where

$$A(z) := \sum_{k\geqslant 0} A_k(z), \text{ with } A_k(z) := \prod_{1\leqslant j\leqslant k} \big(e^{jz}-1\big), \tag{3.1}$$

which is used as the running and prototypical example of our analytic approach. The sequence $\{n!a_n\}_{n\geqslant 0}$ equals A158690 and can be generated, in addition to (1.7), by many different forms (see [8, 2]), showing partly the diversity and structural richness of the sequence

$$\begin{aligned} A(z) &= \sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \big(1-e^{-(2j-1)z}\big) \\ &= \sum_{k\geqslant 0} e^{-(k+1)z} \prod_{1\leqslant j\leqslant k} \big(1-e^{-2jz}\big) \\ &= \sum_{k\geqslant 0} e^{(2k+1)z} \prod_{1\leqslant j\leqslant 2k} \big(e^{jz}-1\big) \\ &= \frac{1}{2}\Big(1 + \sum_{k\geqslant 0} e^{(k+1)z} \prod_{1\leqslant j\leqslant k} \big(e^{jz}-1\big)\Big). \end{aligned}$$

Among these series forms, we work on (3.1) because it is simpler and the terms in the summation contain only positive Taylor coefficients.

Theorem 5. *As n tends to infinity, the sequence A158690 satisfies*

$$a_n := [z^n]A(z) \simeq c\rho^n n^{n+\frac{1}{2}}, \text{ with } (c,\rho) = \Big(\tfrac{12}{\pi^{3/2}}, \tfrac{12}{e\pi^2}\Big). \tag{3.2}$$

Remark 1. *Alternatively, (3.2) can be written as*

$$a_n \simeq c\rho^n n!, \text{ with } (c,\rho) = \Big(\tfrac{6\sqrt{2}}{\pi^2}, \tfrac{12}{\pi^2}\Big).$$

Since $\frac{12}{e\pi^2} < 1 < \frac{12}{\pi^2}$, we see that $n! \ll a_n \ll n^n$. Indeed, with

$$a_{n,k} := [z^n]A_k(z),$$

we have $a_{n,n} = n!$, and it can be proved that

$$a_{n,k} \sim \begin{cases} \dfrac{n^{n-k}}{(n-k)!4^{n-k}}\, n!, & \text{if } 0 \leqslant n-k = o(\sqrt{n}), \\ \dfrac{1}{n!}\Big(\dfrac{k(k+1)}{2}\Big)^n, & \text{if } 1 \leqslant k = o(\sqrt{n}), \end{cases}$$

so that the major contribution to a_n does not come from the ranges when k is either very small compared to n or very close to n.

Our approach consists simply in computing the asymptotics of $a_{n,k}$ by the saddle-point method (see [20]) for each $1 \leqslant k < n$, and then summing $a_{n,k}$ over all k (in turn involving another application of saddle-point method); indeed, due to high concentration near the maximum, only a small neighborhood of k near μn, $\mu := \frac{12}{\pi^2}\log 2 \approx 0.84$, will contribute to the dominant asymptotics (3.2). Thus we are in the context of a double saddle-point method.

More precisely, we begin with the expression

$$a_n = \sum_{1\leqslant k\leqslant n} a_{n,k} = \sum_{1\leqslant k\leqslant n} \frac{r^{-n}}{2\pi i}\int_{-\pi}^{\pi} e^{-in\theta} A_k(re^{i\theta})\,\mathrm{d}\theta,$$

and follow the procedures outlined below.

- Find the positive solution pair (k, r) of the equations $(\partial_k r^{-n} A_k(r), \partial_r r^{-n} A_k(r)) = (0, 0)$, so as to identify the terms $a_{n,k}$ reaching the maximum modulus for each fixed n; see Lemma 11.
- Once the range of $k \sim \mu n$ is identified, show, by simple saddle-point bound for Taylor coefficients, that the contribution to a_n of $a_{n,k}$ from the range $|k-\mu n| \geqslant n^{\frac{5}{8}}$ is asymptotically negligible; see Proposition 15.
- In the central range $|k - \mu n| \leqslant n^{\frac{5}{8}}$, the integral $\int_{n^{-\frac{3}{8}}\leqslant|\theta|\leqslant\pi}$ is asymptotically negligible; see Proposition 17.
- Then inside the ranges $|k-\mu n| \leqslant n^{\frac{5}{8}}$ and $\int_{|\theta|\leqslant n^{-\frac{3}{8}}}$, compute the asymptotic approximation (3.2) by local expansions and term-by-term integration; see Section 3.5.
- These procedures can be refined to get finer expansions if desired.

For all these purposes, it turns out that a precise asymptotic approximation to $\log A_k(r)$ will largely simply the analysis. Since we will also need asymptotics of the derivatives of $\log A_k(r)$, we propose a complex-variable version so as to avoid repeated use of the Euler-Maclaurin formula.

3.1. **Euler-Maclaurin formula and asymptotic expansions.** We apply the Euler-Maclaurin formula to approximate the various sums encountered in this paper, which for completeness is included as follows.

Lemma 6 (Euler-Maclaurin formula)**.** *Assume that φ is m-times continuously differentiable over the interval $[a, b]$, $m \geqslant 1$. Then*

$$\sum_{j=a+1}^{b} \varphi(j) = \int_a^b \varphi(t)\,\mathrm{d}t + \frac{\varphi(b)-\varphi(a)}{2} + \sum_{\ell=1}^{\lfloor m/2\rfloor} \frac{B_{2\ell}}{(2\ell)!}\big(\varphi^{(2\ell-1)}(b) - \varphi^{(2\ell-1)}(a)\big) + \frac{(-1)^{m+1}}{m!}\int_a^b \varphi^{(m)}(t)B_m(\{t\})\mathrm{d}t, \tag{3.3}$$

where $\{x\}$ denotes the fractional part of x, the B_ℓ's and the $B_n(t)$'s are Bernoulli numbers and polynomials, respectively.

When φ is infinitely differentiable (which is the case for all functions considered in this paper), we can push the expansion to any $m > 0$ depending on the required error, making the error term under control.

The following expansion is crucial in our saddle-point analysis. Let

$$L_k(z) := \log A_k(z) = \sum_{1\leqslant j\leqslant k} \log\big(e^{jz}-1\big).$$

Proposition 7. *For $k\to\infty$, we have*

$$\begin{aligned} L_k(z) &= k\log\big(e^{kz}-1\big) - \frac{I(kz)}{z} + \frac{1}{2}\log\frac{2\pi\big(e^{kz}-1\big)}{z} \\ &\quad + \frac{z\big(e^{kz}+1\big)}{24\big(e^{kz}-1\big)} + \sum_{2\leqslant j<m} \frac{B_{2j}}{(2j)!}\cdot\frac{z^{2j-1}e^{-kz}E_{2j-2}\big(e^{-kz}\big)}{\big(1-e^{-kz}\big)^{2j-1}} \\ &\quad + O\big(k^{1-2m}+z^{2m-1}\big), \end{aligned} \tag{3.4}$$

uniformly for $k|z|\leqslant 2\pi-\varepsilon$ when $|\arg z|\leqslant \pi-\varepsilon$, where

$$I(z) := \int_0^z \frac{t}{1-e^{-t}}\,\mathrm{d}t,$$

and $E_n(x) = \sum_{0\leqslant j<n} \left\langle {n \atop j} \right\rangle x^j$ denote the polynomials of Eulerian numbers.

Proof. For simplicity and for later use, we compute only the first few terms by working out $m=2$, the general form following from the relation

$$\partial_z^m \log\big(e^{xz}-1\big) = (-1)^{m-1}\frac{x^m e^{-xz}E_{m-1}\big(e^{-xz}\big)}{\big(1-e^{-xz}\big)^m} \quad (m\geqslant 2);$$

see [40] for similar analysis.

Since $\log\big(e^{jz}-1\big)$ is undefined at $j=0$, we split the sum into two parts:

$$L_k(z) = \log k! - \sum_{1\leqslant j\leqslant k}\log\frac{j}{e^{jz}-1}.$$

By the Euler-Maclaurin formula (3.3), we find that

$$\begin{aligned} \sum_{1\leqslant j\leqslant k}\log\frac{j}{e^{jz}-1} &= \int_0^k \log\frac{x}{e^{xz}-1}\,\mathrm{d}x + \frac{1}{2}\log\frac{kz}{e^{kz}-1} \\ &\quad + \frac{1}{12}\left(\frac{1}{k}+\frac{z}{2}-\frac{z}{1-e^{-kz}}\right) + O\big(k^{-2}+|z|^2\big). \end{aligned}$$

By an integration by parts, we see that

$$\int_0^k \log\frac{x}{e^{xz}-1}\,\mathrm{d}x = k\log\frac{k}{e^{kz}-1} - k + \frac{I(kz)}{z}.$$

The first few terms of (3.4) then follow from this and Stirling's formula for $\log k!$.

For the error term, we have, by (3.6) with $m=2$ and $B_2(x)=x^2-x+\frac{1}{6}$,

$$R_2 := \int_0^k \left(\frac{z^2e^{xz}}{\big(e^{xz}-1\big)^2} - \frac{1}{x^2}\right)\left(\{x\}^2-\{x\}+\tfrac{1}{6}\right)\mathrm{d}x.$$

Since $k|z| \leqslant 1$, we see that

$$R_2 = O\left(\int_0^{1/|z|} |z|^2 \,\mathrm{d}x\right) = O(|z|).$$

On the other hand, if $1 \leqslant k|z| \leqslant 2\pi - \varepsilon$, then

$$R_2 = O\left(|z| + \int_{1/|z|}^{k} \left(\frac{|z|^2 e^{\Re(xz)}}{|e^{xz}-1|^2} + \frac{1}{x^2}\right) \mathrm{d}x\right) = O\big(|z| + k^{-1}\big),$$

as required, where $\Re(z)$ denotes the real part of z. This proves (3.4). □

Note that

$$I(z) = \int_0^z \frac{t}{1-e^{-t}}\,\mathrm{d}t = \frac{z^2}{2} + \operatorname{dilog}\big(e^{-z}\big),$$

where $\operatorname{dilog}(1-z) = \sum_{k\geqslant 1} \frac{z^k}{k^2}$ denotes the dilogarithm function. Also

$$\operatorname{dilog}\big(e^{-z}\big) = \sum_{j\geqslant 1} \frac{B_{j-1}}{j!}\, z^j, \quad (|z| < 2\pi),$$

where the B_j's denote the Bernoulli numbers.

The main reason of stating this complex-variable version for $L_k(z)$ is that termwise differentiation with respect to z is allowed by analyticity in compact domain (or Cauchy's integral formula for derivatives), leading to an asymptotic expansion for all higher derivatives of $L_k(z)$; see, e.g., [36, 42]. In this way, we obtain, for example, the following approximations, which will be needed below.

Corollary 8. *Uniformly as $k \to \infty$ and $k|z| \leqslant 2\pi - \varepsilon$ when $|\arg(z)| \leqslant \pi - \varepsilon$,*

$$zL_k'(z) = \sum_{1\leqslant j\leqslant k} \frac{jz}{1-e^{-jz}} = \frac{I(kz)}{z} + \frac{kz-1+e^{-kz}}{2\big(1-e^{-kz}\big)} + O\big(k^{-1} + |z|\big). \tag{3.5}$$

Corollary 9. *Let $m \geqslant 2$. Then*

$$\begin{aligned} z^m L_k^{(m)}(z) &= (-1)^{m-1} z^m \sum_{1\leqslant j\leqslant k} \frac{j^m e^{-jz} E_{m-1}\big(e^{-jz}\big)}{\big(1-e^{-jz}\big)^m} \\ &= z^m \partial_z^{m-1}\left(\frac{I(kz)}{z^2} + \frac{kz-1+e^{-kz}}{2z\big(1-e^{-kz}\big)}\right) + O\big(k^{-1} + |z|\big), \end{aligned} \tag{3.6}$$

uniformly as $k \to \infty$, $k|z| \leqslant 2\pi - \varepsilon$ when $|\arg z| \leqslant \pi - \varepsilon$.

In particular, we see that each $r^m L_k^{(m)}(r)$ is asymptotically of linear order when $kr = O(1)$.

3.2. **Saddle-point method. I: Identifying the central range.** A very simple uniform estimate for $a_{n,k}$ is readily obtained by the saddle-point bound for positive Taylor coefficients (see [20, Sec. VIII.2]).

Lemma 10. *For* $1 \leqslant k < n$

$$a_{n,k} \leqslant r^{-n} A_k(r), \tag{3.7}$$

where $r > 0$ *is chosen to be the saddle-point, namely, the unique positive solution of the equation*

$$rL_k'(r) = \frac{rA_k'(r)}{A_k(r)} = \sum_{1 \leqslant j \leqslant k} \frac{jr}{1 - e^{-jr}} = n. \tag{3.8}$$

Such an r obviously exists for $n \geqslant 1$ and $1 \leqslant k < n$ because $x/(1-e^{-x}) \geqslant 1$ is monotonically increasing with $x \geqslant 0$. Also $r \to \infty$ when $k = o(n)$ and $r \to 0$ when $k \to n$. In particular, $r = 0$ when $k = n$. See Corollary 13 for a more precise description.

The simple bound (3.7) is sufficient to give not only the right factorial term n^n in (3.2) but also the right exponential one $\left(\frac{12}{e\pi^2}\right)^n$; see Lemma 12.

Lemma 11. *For* $1 \leqslant k = qn < n$

$$a_{n,k} = O\left(n^{n+\frac{1}{2}} e^{\phi(q,\varrho)n}\right), \tag{3.9}$$

where

$$\phi(q, \varrho) = -\log \varrho + q \log(e^{q\varrho} - 1) - 1, \tag{3.10}$$

and $\varrho = nr > 0$ *solves the equation*

$$I(q\varrho) = \varrho. \tag{3.11}$$

Proof. We begin with the first-order approximation to $rL_k'(r)$ already derived in (3.5) with the saddle-point $|z| = r = \frac{\varrho}{n}$ and $k = qn$

$$\frac{\varrho}{n} L_k'\left(\frac{\varrho}{n}\right) \sim \frac{n}{\varrho} I(q\varrho),$$

which leads to the approximate saddle-point equation (3.11).

On the other hand, we have, by (3.4),

$$\log\left(r^{-n} A_k(r)\right) = n \log n + n\phi(q, \varrho) + \tfrac{1}{2} \log n + O(1),$$

where ϕ is given by

$$\phi(q, \varrho) = -\log \varrho + q \log(e^{q\varrho} - 1) - \frac{I(q\varrho)}{\varrho}.$$

By (3.11), we then obtain (3.10). $\square$

Lemma 12. *The maximum of* $\phi(q, \varrho)$ *for* $q \in [0, 1]$ *subject to the condition* $I(q\varrho) = \varrho$ *is reached by the pair*

$$(q, \varrho) = (\mu, \xi) := \left(\frac{12}{\pi^2} \log 2, \frac{\pi^2}{12}\right), \tag{3.12}$$

with the maximum value $\phi(\mu, \xi) = -\log \frac{\pi^2}{12} - 1$.

Proof. We solve the system of equations $(\partial_q\phi, \partial_\varrho\phi) = (0, 0)$ with positive solution, or, equivalently,

$$\begin{cases} \log\big(e^{q\varrho} - 1\big) = 0, \\ I(q\varrho) = \varrho. \end{cases}$$

From the first equation, we have $q\varrho = \log 2$. Then by the relation

$$I(\log 2) = \int_0^{\log 2} \frac{t}{1 - e^{-t}}\, \mathrm{d}t = \frac{\pi^2}{12},$$

we obtain the solution pair (3.12). This is one of the sources of the ubiquitous factor $\frac{\pi^2}{6}$ in this paper. Now, by viewing $w = w(q)$ as a function of q, we see that, when (q, w) satisfies the condition $I(qw) = w$,

$$\partial_q^2\phi(q, w) = \frac{w}{1 - q^2 w - e^{-qw}}.$$

We now prove that $\partial_q^2\phi(q, w) < 0$ for all pairs (q, w) such that $I(qw) = w$. First, the function $t \mapsto \frac{t}{1-e^{-t}}$ is motononically increasing for $t \geqslant 0$; then, with $w \neq 0$,

$$w = \int_0^{qw} \frac{t}{1 - e^{-t}}\, \mathrm{d}t < \frac{qw}{1 - e^{-qw}} \cdot qw = \frac{q^2 w^2}{1 - e^{-qw}},$$

implying that

$$w\left(1 - \frac{q^2 w}{1 - e^{-qw}}\right) = \frac{w\big(1 - q^2 w - e^{-qw}\big)}{1 - e^{-qw}} < 0.$$

Thus the function $q \mapsto \partial_q^2\phi(q, w)$ is always negative for all pairs (q, w) such that $I(qw) = w$, showing that $q \mapsto \phi(q, w)$ is concave downward when w satisfies $I(qw) = w$; see Figure 3.1. This proves the lemma.

$\square$

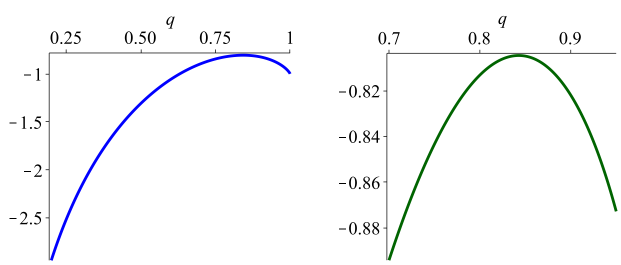


Figure 3.1. The concavity of the function $\phi(q, \varrho)$ when $\varrho = \varrho(q)$ satisfies (3.11) for $q \in [0.2, 1]$ (left) and $q \in [0.7, 0.95]$ (right).

From the equation (3.11), we have the uniform estimate for the saddle-point r in (3.8). The notation $a_n \asymp b_n$ means that the ratio of the two sequences remain bounded and nonzero.

Corollary 13. *The saddle-point r satisfies $r \asymp \frac{n-k}{k^2}$ for $1 \leqslant k < n$.*

Proof. We have

$$I(x) = \begin{cases} x + \frac{1}{4}x^2 + O(x^3), \text{as } x \to 0, \\ \frac{1}{2}x^2 + \frac{\pi^2}{6} - xe^{-x} + O(e^{-x}), \text{as } x \to \infty. \end{cases}$$

If $k = o(n)$, then $q\varrho = kr \to \infty$ and the left-hand side of (3.11) is asymptotic to $\frac{1}{2}(kr)^2$, and we then have $r \sim \frac{2n}{k^2}$. On the other hand, if $n - k = o(n)$ or $kr \to 0$, then the left-hand side of (3.11) is asymptotic to $kr + \frac{1}{4}(kr)^2$, so we get $r \sim \frac{4(n-k)}{k^2}$. In either case, we have $r \asymp \frac{n-k}{k^2}$. Note that $r \to 0$ iff $\frac{k}{\sqrt{n}} \to \infty$. □

3.3. **Saddle-point method. II: Negligibility of summands outside the central range.** Define

$$\sigma := \pi^{-2}\sqrt{6\big(24(\log 2)^2 - \pi^2\big)} \approx 0.31988. \tag{3.13}$$

Lemma 14. *Write $k = \mu n + x\sigma\sqrt{n}$. Then uniformly for $x = o\big(n^{\frac{1}{6}}\big)$,*

$$a_{n,k} = O\big(\rho^n n^{n+\frac{1}{2}} e^{-\frac{1}{2}x^2}\big), \quad \textit{with} \quad \rho = \frac{12}{e\pi^2}. \tag{3.14}$$

Proof. Assume first

$$q = \mu + \frac{\sigma x}{\sqrt{n}}, \tag{3.15}$$

where μ is already determined in Lemma 12 but the value of σ given in (3.13) has remained unknown (and will be specified by the following procedure). Substituting this q into the saddle-point equation (3.8), as approximated by (3.5), and solving asymptotically for ϱ, we then obtain

$$\varrho = \xi + \frac{\xi_1 x}{\sqrt{n}} + \frac{\xi_2 + \xi_3 x^2}{n} + O\Big(\frac{|x| + |x|^3}{n^{\frac{3}{2}}}\Big), \tag{3.16}$$

where, with $\tau = 2(\log 2)^2 - \frac{\pi^2}{12}$

$$\xi_1 = -\frac{\pi^4 \log 2}{72\,\tau}, \quad \xi_2 = -\frac{\pi^4(2\log 2 - 1)}{288\,\tau} \quad \text{and} \quad \xi_3 = \frac{\pi^6(288\tau^2 + (\log 2)\pi^4 + 24\pi^2\tau - \pi^4)}{248832\,\tau^3}. \tag{3.17}$$

Then we substitute the expansions (3.15) and (3.16) into (3.10), giving

$$\phi(q,\varrho) = -\log\frac{\pi^2}{12} - 1 + \frac{1}{n}\left(\frac{1}{2} - \log 2 - \frac{\pi^4\sigma^2x^2}{144\tau}\right) + O\Big(\frac{|x| + |x|^3}{n^{\frac{3}{2}}}\Big).$$

So if we take $\sigma^2 = 72\pi^{-4}\tau$ (which is identical to the expression given in (3.13)), then we see that

$$e^{n\phi(q,\varrho)} = \frac{\sqrt{e}}{2}\left(\frac{12}{e\pi^2}\right)^n e^{-\frac{1}{2}x^2}\left(1 + O\Big(\frac{|x| + |x|^3}{\sqrt{n}}\Big)\right),$$

uniformly for $x = o\big(n^{\frac{1}{6}}\big)$. This, together with (3.9) and Lemma 12, proves (3.14). □

Proposition 15. *Let*

$$k_\pm := \mu n \pm \sqrt{2}\sigma n^{\frac{5}{8}}, \tag{3.18}$$

where μ and σ are given in (3.12) *and* (3.13)*, respectively. Then, with* $\rho = \frac{12}{e\pi^2}$,

$$\left(\sum_{1\leqslant k<k_-} + \sum_{k_+<k\leqslant n}\right) a_{n,k} = O\Big(\rho^n n^{n+\frac{3}{2}} e^{-n^{\frac{1}{4}}}\Big). \tag{3.19}$$

Proof. By monotonicity of $\phi(q, w)$ (see Lemma 12),

$$\left(\sum_{1\leqslant k<k_-} + \sum_{k_+<k\leqslant n}\right) a_{n,k} = O(na_{n,k_-} + na_{n,k_+}).$$

Then (3.19) follows from (3.14) with $x = \sqrt{2}\, n^{\frac{1}{8}}$. □

3.4. **Saddle-point method. III: Negligibility of integrals away from zero.** We now show that in the remaining sum ($k_\pm$ defined in (3.18))

$$\sum_{k_-\leqslant k\leqslant k_+} \frac{r^{-n}}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} A_k(re^{i\theta})\,\mathrm{d}\theta,$$

the integral over the range $\theta_0 \leqslant |\theta| \leqslant \pi$, $\theta_0 := 6n^{-\frac{3}{8}}$ is asymptotically negligible. Such a θ_0 is always chosen so that $n\theta_0^2 \to \infty$ and $n\theta_0^3 \to 0$; see [20]. We begin with a uniform bound for $|A_k(z)|$.

Lemma 16. *Let* $\theta := \arg(z)$*. Then, uniformly for* $|z| > 0$ *and* $|\theta| \leqslant \pi$*,*

$$|A_k(z)| \leqslant A_k(|z|) \exp\left(-\frac{k(k+1)|z|\,\theta^2}{2\pi^2}\right), \qquad (k = 1, 2, \dots). \tag{3.20}$$

Proof. The uniform bound (3.20) is a direct consequence of the inequality (see [38, Appendix])

$$\left|e^z - 1\right| \leqslant \big(e^{|z|} - 1\big)e^{-|z|\theta^2/\pi^2}, \qquad (|\theta| \leqslant \pi). \tag{3.21}$$

This is proved as follows. First

$$\begin{aligned}
\left|e^z - 1\right| &= \left|e^{\frac{1}{2}z}\right|\left|e^{\frac{1}{2}z} - e^{-\frac{1}{2}z}\right| \\
&\leqslant e^{\frac{1}{2}|z|\cos\theta}\big(e^{\frac{1}{2}|z|} - e^{-\frac{1}{2}|z|}\big) \\
&= \big(e^{|z|} - 1\big)e^{-\frac{1}{2}|z|(1-\cos\theta)},
\end{aligned}$$

where the inequality results from the fact that $[t^n]\big(e^t - e^{-t}\big) \geqslant 0$ for all $n \geqslant 0$. Then (3.21) follows from the elementary inequality $1 - \cos\theta \geqslant \frac{2}{\pi^2}\theta^2$ for $|\theta| \leqslant \pi$. □

Proposition 17. *Define* $k_\pm$ *as in* (3.18) *and* $\theta_0 := 6n^{-\frac{3}{8}}$*. Then, with* $\rho = \frac{12}{e\pi^2}$,

$$\sum_{k_-\leqslant k\leqslant k_+} \frac{r^{-n}}{2\pi} \int_{\theta_0\leqslant|\theta|\leqslant\pi} e^{-in\theta} A_k(re^{i\theta})\,\mathrm{d}\theta = O\big(\rho^n n^{n-\frac{1}{8}} e^{-n^{\frac{1}{4}}}\big). \tag{3.22}$$

Proof. By (3.20) with $z = re^{i\theta}$,

$$\sum_{k_-\leqslant k\leqslant k_+} \frac{r^{-n}}{2\pi}\int_{\theta_0\leqslant|\theta|\leqslant\pi} e^{-in\theta}A_k(re^{i\theta})\,\mathrm{d}\theta = O\left(\sum_{k_-\leqslant k\leqslant k_+} r^{-n}A_k(r)\int_{\theta_0}^{\infty} e^{-\frac{k^2r\theta^2}{2\pi^2}}\,\mathrm{d}\theta\right).$$

Now, with $k \sim \mu n$ ($k_- \leqslant k \leqslant k_+$) and $rn \sim \xi$ (see (3.12)), we then have

$$\begin{aligned}\int_{\theta_0}^{\infty} e^{-\frac{k^2r\theta^2}{2\pi^2}}\,\mathrm{d}\theta &= O\left(\frac{n^{\frac{3}{8}}}{k^2r}e^{-\frac{k^2r}{2\pi^2n^{3/4}}}\right)\\ &= O\left(n^{-\frac{5}{8}}e^{-\frac{216(\log 2)^2}{\pi^4}(1+o(1))n^{\frac{1}{4}}}\right).\end{aligned}$$

Note that $\frac{216(\log 2)^2}{\pi^4} \approx 1.065\cdots > 1$. Then (3.22) follows from (3.14). $\square$

3.5. **Saddle-point method. IV: Proof of Theorem 5.** From the two estimates (3.19) and (3.22), we have, with $\rho = \frac{12}{e\pi^2}$,

$$a_n = \sum_{k_-\leqslant k\leqslant k_+} r^{-n}A_k(r)J_I + O\Big(\rho^n n^{n+\frac{3}{2}}e^{-n^{\frac{1}{4}}}\Big) \tag{3.23}$$

where ($\theta_0 = 6n^{-\frac{3}{8}}$)

$$J_I := \frac{1}{2\pi}\int_{-\theta_0}^{\theta_0} e^{-in\theta}\frac{A_k(re^{i\theta})}{A_k(r)}\,\mathrm{d}\theta.$$

We begin by evaluating asymptotically the integral J_I.

Lemma 18. *If $k = \mu n + x\sigma\sqrt{n}$, where μ and σ are given in (3.12) and (3.13), respectively, then*

$$J_I \simeq \frac{\sqrt{3}}{\pi^{3/2}\sigma}n^{-\frac{1}{2}}, \tag{3.24}$$

uniformly for $x = o\big(n^{\frac{1}{6}}\big)$.

Proof. Expand $L_k(re^{i\theta})$ in θ:

$$\log\frac{A_k\big(re^{i\theta}\big)}{A_k(r)} = L_k\big(re^{i\theta}\big) - L_k(r) := \sum_{j\geqslant 1}\frac{\upsilon_j(r)}{j!}(i\theta)^j.$$

First of all, $\upsilon_1(r) = \frac{rA_k'(r)}{A_k(r)} = rL_k'(r) = n$ by our choice of r. Then by (3.6) with $q = \frac{k}{n}$ and $\varrho = nr$ satisfying (3.15) and (3.16), we obtain

$$\upsilon_2(r) = r^2L_k''(r) + rL_k'(r) = \Big(\frac{24}{\pi^2}(\log 2)^2 - 1\Big)n + O(1) = \frac{\pi^2}{6}\sigma^2 n + O(1).$$

Furthermore, each $\upsilon_j(r) \asymp n$ by (3.6) when $k_- \leqslant k \leqslant k_+$. Thus $\upsilon_j(r)\theta_0^j \to 0$ for $j = 3, 4, \ldots,$ and we then obtain

$$\begin{aligned}
J_I &= \frac{1}{2\pi}\int_{-\theta_0}^{\theta_0} e^{-\frac{1}{2}\upsilon_2(r)\theta^2 - \frac{1}{6}\upsilon_3(r)i\theta^3 + O(n\theta^4)}\,\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_{-\theta_0}^{\theta_0} e^{-\frac{1}{2}\upsilon_2(r)\theta^2}\left(1 - \tfrac{1}{6}\upsilon_3(r)i\theta^3 + O(n\theta^4 + n\theta^6)\right)\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\upsilon_2(r)\theta^2}\left(1 - \tfrac{1}{6}\upsilon_3(r)i\theta^3 + O(n\theta^4 + n\theta^6)\right)\mathrm{d}\theta \\
&\quad + O\left(\upsilon_2^{-1} n^{\frac{3}{8}} e^{-18\upsilon_2(r)n^{-\frac{3}{4}}}\right) \\
&= \frac{1}{\sqrt{2\pi\upsilon_2(r)}}\left(1 + O\left(n^{-1}\right)\right) + O\left(n^{-\frac{5}{8}} e^{-3\pi^2\sigma^2 n^{\frac{1}{4}}}\right) \\
&\simeq \frac{\sqrt{3}}{\pi^{3/2}\sigma} n^{-\frac{1}{2}}.
\end{aligned}$$

This proves (3.24). □

Proof of Theorem 5. With (3.23) and (3.24) available, we can now complete the proof of Theorem5 by deriving the finer expansion

$$r^{-n}A_k(r) = c_0\rho^n n^{n+\frac{1}{2}} e^{-\frac{1}{2}x^2}\left(1 + \frac{g_1(x)}{\sqrt{n}} + O\left(n^{-1}(1+x^6)\right)\right),$$

where $(c_0, \rho) = \left(\sqrt{\frac{24}{\pi}}, \frac{12}{e\pi^2}\right)$ and $g_1(x)$ is an odd polynomial in x of degree three (whose expression being immaterial here). It follows that

$$\begin{aligned}
a_{n,k} &= [z^n]A_k(z) = \frac{\sqrt{3}}{\pi^{3/2}\sigma} n^{-\frac{1}{2}} r^{-n} A_k(r)\left(1 + O\left(n^{-1}\right)\right) \\
&= \frac{c_1}{\sigma}\rho^n n^n e^{-\frac{1}{2}x^2}\left(1 + \frac{g_1(x)}{\sqrt{n}} + O\left(n^{-1}(1+x^6)\right)\right),
\end{aligned} \tag{3.25}$$

uniformly for $k_- \leqslant k \leqslant k_+$, where $(c_1, \rho) = \left(\frac{\sqrt{72}}{\pi^2}, \frac{12}{e\pi^2}\right)$, where σ is given in (3.13).

From this and the two estimates (3.19) and (3.22), we obtain

$$a_n = \frac{12}{\pi^{3/2}}\rho^n n^n \sum_{k_- \leqslant k \leqslant k_+} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}\,\sigma}\left(1 + \frac{g_1(x)}{\sqrt{n}} + O\left(n^{-1}(1+x^6)\right)\right) + O\left(\rho^n n^{n+\frac{3}{2}} e^{-n^{\frac{1}{4}}}\right),$$

from which we deduce (3.2) by approximating the sum by integral. □

Remark 2. *We have proved more than the asymptotic estimate (3.2); indeed, if we define the random variable X_n by*

$$\mathbb{P}(X_n = k) = \frac{[z^n]A_k(z)}{[z^n]A(z)} \qquad (1 \leqslant k \leqslant n),$$

then our asymptotic expansions (3.25) *and* (4.1) *imply obviously the local limit theorem (in the form of moderate deviations):*

$$\mathbb{P}(X_n = \mu n + x\sigma\sqrt{n}) = \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi\sigma^2 n}}\left(1 + O\Big(\frac{|x| + |x|^3}{\sqrt{n}}\Big)\right),$$

uniformly for $x = o\big(n^{\frac{1}{6}}\big)$.

4. Asymptotic expansions and change of variables

We examine briefly in this section two different ways to obtain asymptotics expansions for $a_n = [z^n]A(z)$ as defined in (3.2), and then show how an argument based on change of variables leads to expansions for the coefficients under different parametrization of the underlying function.

The first approach to deriving an expansion of the form

$$[z^n]A(z) = c\rho^n n^{n+\frac{1}{2}}\left(1 + \sum_{1\leqslant j<m} \nu_j n^{-j} + O\big(n^{-m}\big)\right), \tag{4.1}$$

for some computable coefficients ν_j, is now straightforward following the same analysis detailed in the previous section. It consists in first computing an asymptotic expansion for $a_{n,k}$, which is of the form

$$a_{n,k} = \frac{c_1}{\sigma}\,\rho^n n^n e^{-\frac{1}{2}x^2}\left(1 + \sum_{1\leqslant j<m} \frac{g_j(x)}{n^{\frac{1}{2}j}} + O\big(n^{-\frac{1}{2}m}\big)\right), \text{ with } (c,\rho) = \big(\tfrac{12}{\pi^{3/2}}, \tfrac{12}{e\pi^2}\big)$$

uniformly for $k = \mu n + x\sigma\sqrt{n}$, $x = o(n^{\frac{1}{6}})$, where $g_j(x)$ is a computable polynomial in x of degree $3j$ and contains only powers of x with the same parity as j. From this we can then deduce (4.1) by approximating the sum by an integral and extending the integration range to $\pm\infty$. We omit the messy details as they are more or less standard and all procedures can be readily coded in symbolic computation softwares.

4.1. An asymptotic expansion via Dirichlet series.

For more methodological interest, we sketch here another approach, based on that used in [8], to obtaining asymptotic expansions for a_n when more information is available.

Proposition 19. *The sequence* a_n *in* (3.2) *satisfies the asymptotic expansion*

$$a_n = c\rho^n\, n!\Big(1 + \sum_{1\leqslant j<m} \frac{c_j}{n\cdots(n-j+1)} + O\big(n^{-m}\big)\Big), \tag{4.2}$$

for $m \geqslant 2$, *where* $(c,\rho) = \big(\frac{6\sqrt{2}}{\pi^2}, \frac{12}{\pi^2}\big)$ *and* $c_j = \frac{1}{j!}\big(-\frac{\pi^2}{288}\big)^j$ *for* $j \geqslant 1$.

In particular,

$$a_n = c\rho^n n!\left(1 - \frac{\pi^2}{288n} + \frac{\pi^4}{165888\, n(n-1)} - + \cdots\right).$$

The very simple form of the coefficients c_j naturally suggests the following approximation:

$$a_n = c\rho^n n! e^{-\frac{\pi^2}{288n}}\left(1 + O\big(n^{-3}\big)\right),$$

which has obvious numerical advantages; see Figure 4.1 for a few graphical renderings of the numerical goodness of (4.2).

Proof. We begin with (1.7). As in [8], we define the Dirichlet series

$$D(s) := \sum_{n\geqslant 1} n^{-s}[q^{n-1}]R\big(q^{24}\big),$$

which converges absolutely in $\Re(s) > 1$ and can be analytically continued into the whole s-plane. Together with Mellin transform techniques, we now have the two relations [8]

$$\begin{cases} a_n = \dfrac{(-1)^n}{2}[z^n]R\big(e^{-z}\big), \\ b_n := [z^n]e^{-\frac{1}{24}z}R\big(e^{-z}\big) = \dfrac{(-1)^n D(-n)}{n!24^n}. \end{cases}$$

Then, by the functional equation derived in [10], one has

$$D(-n) = c_0\rho_0^n n!^2\left(1 + O\big(23^{-n}\big)\right), \quad \text{with } (c_0, \rho_0) = \big(\tfrac{12\sqrt{2}}{\pi^2}, \tfrac{288}{\pi^2}\big),$$

for large n. This implies that

$$b_n = c_0(-1)^n\rho^n n!\left(1 + O\big(23^{-n}\big)\right), \quad \text{with } (c_0, \rho) = \big(\tfrac{12\sqrt{2}}{\pi^2}, \tfrac{12}{\pi^2}\big).$$

From this, we have

$$\frac{b_{n-j}}{b_n} = \frac{(-\rho)^{-j}}{n\cdots(n-j+1)}\left(1 + O\big(23^{-n}\big)\right) \quad (j = 0, 1, \dots), \tag{4.3}$$

implying that the partial sum

$$a_n = \frac{(-1)^n}{2}b_n\sum_{0\leqslant j\leqslant n}\frac{b_{n-j}}{j!24^j b_n}$$

is itself an asymptotic expansion. In this way, we obtain (4.2). □

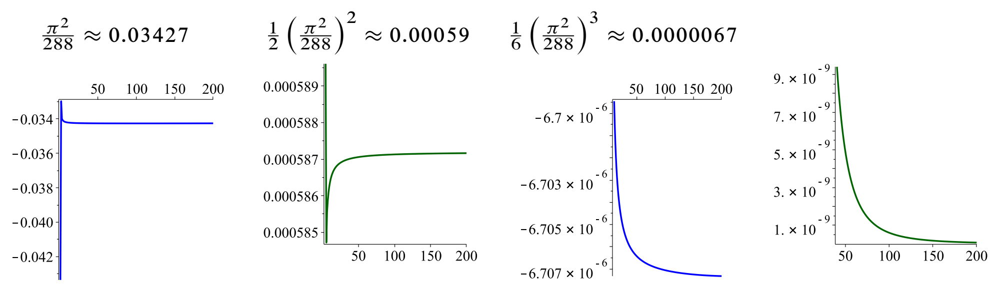


Figure 4.1. The approximation of c_ℓ by using (4.2) for $n \leqslant 200$, namely, $n\cdots(n-\ell+1)\left(\frac{a_n}{c\rho^n n!} - \sum_{0\leqslant j<\ell}\frac{c_j}{n\cdots(n-j+1)}\right)$ for $\ell = 1, 2, 3$ (the first three plots); the last plot corresponds to the difference $\frac{a_n}{c\rho^n n!}e^{\frac{\pi^2}{288n}} - 1$. Here $c = \frac{6\sqrt{2}}{\pi^2}$ and $\rho = \frac{12}{\pi^2}$.

4.2. **From $[z^n]R(e^z)$ to $[z^n]R(1+z)$ by a change of variables.** We sketch here a different technique to derive the asymptotic expansion (1.5) for $[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big((1+z)^j-1\big)$ from that (4.2) for a_n. The original approach by Zagier in [44] and then adapted in [8] relies on the asymtotics of the Stirling numbers of the first kind. We give a direct approach via change of variables, which has the advantages of being easily codable and widely applicable in more general contexts; see Sections 5 and 8.

Define R as in (1.6). Since

$$R(q) = 2\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left(q^j-1\right)$$

is true to infinite order at every root of unity which includes the case $q=1$ (see [10]), by the change of variables $1+z=e^y$, we have,

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big((1+z)^j-1\big) = \frac{1}{2}[z^n]R(1+z) = [y^n]g(y)\big(e^{\frac{1}{24}y}R\big(e^y\big)\big),$$

where

$$g(y) := \frac{1}{2}\left(\frac{y}{e^y-1}\right)^{n+1} e^{\frac{23}{24}y} = \frac{1}{2}\exp\left(-\frac{n}{2}y+\frac{11}{24}y-(n+1)\sum_{j\geqslant 1}\frac{B_{2j}}{2j\cdot(2j)!}\,y^{2j}\right),$$

for small y. Since b_n (see (4.3)) grows factorially with n, and the Taylor coefficients of $g(y)$ is small when compared to b_n, we expand g at $y=\eta$, where η is small and to be determined soon, and then carry out term by term extraction of the coefficients, yielding

$$\begin{aligned}[y^n]g(y)\big(e^{\frac{1}{24}y}R\big(e^y\big)\big) &= \sum_{j\geqslant 0}\frac{g^{(j)}(\eta)}{j!}[y^n](y-\eta)^j e^{\frac{1}{24}y}R\big(e^y\big)\\ &= g(\eta)\bar{b}_n + g'(\eta)\big(\bar{b}_{n-1}-\eta\bar{b}_n\big)+\cdots,\end{aligned}$$

where $\bar{b}_n := (-1)^n b_n = [y^n]e^{\frac{1}{24}y}R\big(e^y\big)$. so if we take (see (4.3))

$$\eta = \frac{\bar{b}_{n-1}}{\bar{b}_n} = \frac{\pi^2}{12n}\left(1+O\big(23^{-n}\big)\right),$$

then the terms involving $g'(\eta)$ become zero, and we have

$$[y^n]\big(e^{\frac{1}{24}y}R\big(e^y\big)\big)g(y) = g(\eta)\bar{b}_n\left(1+\frac{g''(\eta)}{2g(\eta)}\left(\frac{\bar{b}_{n-2}}{\bar{b}_n}-\frac{\bar{b}_{n-1}^2}{\bar{b}_n^2}\right)+\cdots\right).$$

In general, by estimating the Taylor remainders, we deduce the expansion

$$[y^n]\big(e^{\frac{1}{24}y}R\big(e^y\big)\big)g(y) = g(\eta)\bar{b}_n\left(1+\sum_{2\leqslant j\leqslant 2m}\frac{g^{(j)}(\eta)}{j!g(\eta)}H_j(n)+O\big(n^{-m-1}\big)\right),$$

for $m\geqslant 1$, where the general terms are of order $n^{\lceil\frac{1}{2}j\rceil}$ because $g^{(j)}(\eta)=O(n^j)$ and

$$H_j(n) := \sum_{0\leqslant \ell\leqslant j}\binom{j}{\ell}\left(-\frac{\pi^2}{12n}\right)^{j-\ell}\frac{\bar{b}_{n-\ell}}{\bar{b}_n} = \left(\frac{\pi^2}{12}\right)^j\sum_{0\leqslant\ell\leqslant j}\binom{j}{\ell}\frac{(-1)^{j-\ell}(n-\ell)!}{n^{j-\ell}n!}\left(1+O\big(23^{-n}\big)\right),$$

which decays in the order $n^{-j-\lceil\frac{1}{2}j\rceil}$. In this way, we obtain

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big((1+z)^j-1\big)=c\rho^n\, n!\left(1+\sum_{1\leqslant j<m}\frac{c_j}{n^j}+O\big(n^{-m}\big)\right), \tag{4.4}$$

where $(c,\rho)=\big(\frac{6\sqrt{2}}{\pi^2}\,e^{-\frac{\pi^2}{24}},\frac{12}{\pi^2}\big)$ and

$$\begin{aligned}
c_1 &= \frac{\pi^2(\pi^2+66)}{1728}\approx 0.43333\\
c_2 &= \frac{\pi^4(\pi^4-12\pi^2-3420)}{5971968}\approx -0.05612\\
c_3 &= -\frac{\pi^4(95\pi^8+9360\pi^6-232416\pi^4-27051840\pi^2+709171200)}{1238347284480}\approx -0.03378.
\end{aligned}$$

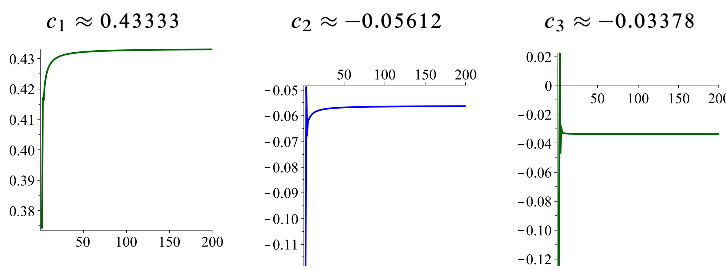


Figure 4.2. The approximation of c_ℓ by using (4.4) for $n\leqslant 200$ and for $\ell=1,2,3$.

5. A framework for matrices with 1s

We consider in this section generating functions of the form

$$\sum_{k\geqslant 0} d(z)^{k+\omega_0}\prod_{1\leqslant j\leqslant k}\big(e(z)^{j+\omega}-1\big)^{\alpha}, \tag{5.1}$$

for $\alpha\in\mathbb{Z}^+$ and $\omega_0,\omega\in\mathbb{C}$, where $d(z)$ and $e(z)$ are formal power series satisfying $d(0)>0$, $e(0)=1$ and $e'(0)\neq 0$. Then we discuss applications to generalized Fishburn matrices and some OEIS sequences.

Our approach consists in examining first the asymptotics of the simpler pattern

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big(e^{(j+\omega)z}-1\big)^{\alpha},$$

for $\alpha\in\mathbb{Z}^+$ and $\omega\in\mathbb{C}$, and follow closely the detailed analysis given in Section 3 for the sequence A158690. Then the extension to (5.1) will rely on the change of variables argument of Section 4.2.

Proposition 20. *For any $\alpha\in\mathbb{Z}^+$ and $\omega\in\mathbb{C}$,*

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big(e^{(j+\omega)z}-1\big)^{\alpha}\simeq c\rho^n n^{n+\alpha\omega+\frac{1}{2}\alpha}, \tag{5.2}$$

uniformly in ω, where the notation "$\simeq$" is defined in (1.4) *and*

$$(c,\rho)=\left(\frac{\sqrt{6}}{\alpha\pi}\left(\frac{2\sqrt{6}}{\sqrt{\alpha\pi}\,\Gamma(1+\omega)}\left(\frac{12}{\alpha\pi^2}\right)^{\omega}\right)^{\alpha},\frac{12}{e\alpha\pi^2}\right).$$

When $\omega\in\mathbb{Z}^-$, the leading constant c is interpreted as zero because of $\Gamma(1+\omega)$ in the denominator, and the right-hand side of (5.2) becomes then a big-O estimate.

Proof. We sketch the major steps for obtaining the dominant term, the error term following from the same procedure with more refined calculations.

- By the Euler-Maclaurin formula (3.3)

$$\begin{aligned}\sum_{1\leqslant j\leqslant k}\log\left(e^{(j+\omega)z}-1\right)&=k\log\left(e^{kz}-1\right)-\frac{I(kz)}{z}+\left(\omega+\frac{1}{2}\right)\log\frac{e^{kz}-1}{z}\\&\quad-\log\Gamma(1+\omega)+\frac{\log 2\pi}{2}+O\left(|\omega|^2\left(k^{-1}+|z|\right)\right),\end{aligned}\tag{5.3}$$

  (compare (3.4)) which holds uniformly $k\to\infty$ and $k|z|\leqslant 2\pi-\varepsilon$ in the sector $|\arg z|\leqslant\pi-\varepsilon$. Here (5.3) holds when $\omega\neq\mathbb{R}^-$. But the asymptotic approximation, by taking the exponential on both sides of (5.3),

$$\begin{aligned}&\prod_{1\leqslant j\leqslant k}\left(e^{(j+\omega)z}-1\right)\\&\quad=\frac{\sqrt{2\pi}}{\Gamma(1+\omega)}\left(\frac{e^{kz}-1}{z}\right)^{\omega+\frac{1}{2}}\left(e^{kz}-1\right)^k e^{-I(kz)/z}\left(1+O\left(|\omega|^2\left(k^{-1}+|z|\right)\right)\right)\end{aligned}$$

  does hold for bounded ω, provided we interpreted the factor $\frac{1}{\Gamma(1+\omega)}$ as zero when $\omega\in\mathbb{Z}^-$.
- The saddle-point equation satisfies asymptotically, by the same differentiation argument used for deriving (3.5),

$$\frac{\alpha}{r}I(kr)+\frac{\alpha}{2}(2\omega+1)\left(\frac{kr}{1-e^{-kr}}-1\right)+O\left(k^{-1}+r\right)=n.$$

  Since the dominant term is independent of ω, we deduce that $k=qn$ with $q\sim\frac{\mu}{\alpha}$ and $rn\sim\alpha\xi$, where $(\mu,\xi)=\left(\frac{12}{\pi^2}\log 2,\frac{\pi^2}{12}\right)$ is the same as in (3.12).
- Observe that for large $k\leqslant n$

$$\prod_{1\leqslant j\leqslant k}\left|e^{(j+\omega)z}-1\right|=O\left(k^{\Re(\omega)}\right)\prod_{1\leqslant j\leqslant k}\left|e^{jz}-1\right|,$$

  when $|z|\asymp n^{-1}$ and $\omega=O(1)$. Then the smallness of the sum

$$\sum_{|k-\frac{\mu}{\alpha}n|\geqslant\sqrt{2}\sigma n^{\frac{5}{8}}}[z^n]\prod_{1\leqslant j\leqslant k}\left(e^{(j+\omega)z}-1\right)^{\alpha},$$

  as well as the corresponding sum of integrals $\sum_{|k-\frac{\mu}{\alpha}n|\leqslant\sqrt{2}\sigma n^{\frac{5}{8}}}\int_{6n^{-\frac{3}{8}}\leqslant|\theta|\leqslant\pi}$ follows from the same bounding techniques used in the proofs of Propositions 15 and 17.

- Inside the central range $\frac{1}{\alpha}k_- \leqslant k \leqslant \frac{1}{k_+}\alpha$, where $k_\pm$ is defined in (3.18), write, as before, $q = \frac{1}{\alpha}\left(\mu + \sigma\frac{x}{\sqrt{n}}\right)$, and solve the saddle-point equation for r, giving

$$rn = \alpha\xi + \frac{\alpha\xi_1 x}{\sqrt{n}} + \frac{\alpha^2\xi_2(1+2\omega) + \alpha\xi_3 x^2}{n} + O\Big(\frac{|x|+|x|^3}{n^{3/2}}\Big), \tag{5.4}$$

  where ξ_i are defined in (3.17).
- We then obtain

$$r^{-n}\prod_{1\leqslant j\leqslant k}\left(e^{(j+\omega)r}-1\right)^\alpha \sim c_0\rho^n n^{n+\alpha(\frac{1}{2}+\omega)},$$

  where

$$(c_0,\rho) = \left(\left(\frac{2\sqrt{6}}{\sqrt{\alpha\pi}\,\Gamma(1+\omega)}\Big(\frac{12}{\alpha\pi^2}\Big)^\omega\right)^\alpha, \frac{12}{e\alpha\pi^2}\right).$$
- The remaining saddle-point analysis is similar to that of Theorem 5.

□

The uniformity in ω will be needed in Section 7.

We now consider the framework (5.1).

Theorem 21. *Assume $\alpha \in \mathbb{Z}^+$ and $\omega_0, \omega \in \mathbb{C}$. For any two formal power series $d(z)$ and $e(z)$ satisfying $d(0) = e(0) = 1$ and $e'(0) \neq 0$, we have*

$$[z^n]\sum_{k\geqslant 0} d(z)^{k+\omega_0}\prod_{1\leqslant j\leqslant k}\left(e(z)^{j+\omega}-1\right)^\alpha \simeq c\rho^n n^{n+\alpha(\frac{1}{2}+\omega)}, \tag{5.5}$$

uniformly for bounded ω_0 and ω, where $d_j := [z^j]d(z)$, $e_j := [z^j]e(z)$, and

$$(c,\rho) = \left(\frac{\sqrt{6}}{\alpha\pi}\left(\frac{2\sqrt{6}}{\sqrt{\alpha\pi}\,\Gamma(1+\omega)}\Big(\frac{12}{\alpha\pi^2}\Big)^\omega\right)^\alpha 2^{\frac{d_1}{e_1}} e^{\frac{\alpha\pi^2}{12}\left(\frac{e_2}{e_1^2}-\frac{1}{2}\right)}, \frac{12e_1}{e\alpha\pi^2}\right). \tag{5.6}$$

The situation when $d(0) \neq 1$ is readily modified. Also the error term can be further refined if needed.

We see that the exponential term depends on α and e_1, the polynomial term on α and ω, and the leading constant c on α, ω, d_1, e_1 and e_2. Furthermore, as far as the dominant asymptotics of the coefficients is concerned, the difference in (5.5) and (5.2) is reflected via the first three terms d_1, e_1, e_2 in the Taylor expansions of $d(z)$ and $e(z)$, but not on ω_0.

Proof. By Cauchy's integral formula

$$a_n := \frac{1}{2\pi i}\oint_{|z|=r_0} z^{-n-1}\sum_{1\leqslant k\leqslant\frac{n}{\alpha}} d(z)^{k+\omega_0}\prod_{1\leqslant j\leqslant k}\left(e(z)^{j+\omega}-1\right)^\alpha \,\mathrm{d}z,$$

where $r_0 > 0$. Without loss of generality, we assume that both $d(z)$ and $e(z)$ are analytic at zero; otherwise, we truncate both formal power series after the nth terms, resulting in two polynomials and thus analytic functions at the origin. Since $e(z) = 1 + e_1 z + \cdots$ with $e_1 \neq 0$, the function is locally invertible and we can make the change of variables $e(z) = e^y$, giving

$$a_n = \frac{1}{2\pi i}\oint_{|y|=r}\psi'(y)\psi(y)^{-n-1}\sum_{1\leqslant k\leqslant\frac{n}{\alpha}} d(\psi(y))^{k+\omega_0}A_k(y)\,\mathrm{d}y,$$

where $A_k(y) := \prod_{1\leqslant j\leqslant k}\big(e^{(j+\omega)y}-1\big)^{\alpha}$ and $\psi(y)$ satisfies $\psi(0)=0$ and $e(\psi(y))=e^y$. In particular,

$$\psi_1=[y]\psi(y)=\tfrac{1}{e_1}\quad\text{and}\quad \psi_2=[y^2]\psi(y)=\tfrac{1}{e_1}\big(\tfrac12-\tfrac{e_2}{e_1^2}\big). \tag{5.7}$$

Observe first that for small $|y|$

$$d(\psi(y))^{k+\omega_0}=\big(1+d_1\psi_1 y+\big(d_1\psi_2+d_2\psi_1^2\big)y^2+\cdots\big)^k;$$

on the other hand, from our saddle-point analysis above, the integration path $|y|=r$ is very close to zero with $r\asymp n^{-1}$, and most contribution to a_n comes from terms with k of linear order, so we see that $d(\psi(y))^k$ is bounded and close to $e^{d_1\psi_1 ky}$ for large n. Similarly, by (5.7),

$$\begin{aligned}\psi'(y)\psi(y)^{-n-1}&=\big(\psi_1+2\psi_2 y+O\big(|y|^2\big)\big)\big(\psi_1 y+\psi_2 y^2+O\big(|y|^3\big)\big)^{-n-1}\\&=e_1^n y^{-n-1}e^{-\frac{\psi_2}{\psi_1}ny}\big(1+O\big(|y|+n|y|^2\big)\big).\end{aligned}$$

Thus the same proof of Theorem 5 extends *mutatis mutandis* to this case, and we then obtain the asymptotic approximation

$$\begin{aligned}a_n=&\sum_{\frac{k_-}{\alpha}\leqslant k\leqslant\frac{k_+}{\alpha}}\frac{1}{2\pi i}\oint_{|y|=r}y^{-n-1}A_k(y)e^{-\frac{\psi_2}{\psi_1}ny+d_1\psi_1 ky}\big(1+O\big(|y|+n|y|^2\big)\big)\,\mathrm{d}y\\&+O\big(\rho^n n^{n+\alpha(\Re(\omega)+\frac12)}e^{-n^{\frac14}}\big),\end{aligned}$$

where $k_\pm$ is given in (3.18) and r satisfies (5.4). Since $q=\frac{k}{n}$ satisfies $q=\frac{1}{\alpha}\big(\mu+\sigma\frac{x}{\sqrt{n}}\big)$, we then deduce (5.5) by noting that

$$e^{-\frac{\psi_2}{\psi_1}nr+d_1\psi_1 kr}=e^{-\frac{\psi_2}{\psi_1}\alpha\xi+d_1\psi_1\mu\xi}\left(1+\frac{\tilde g_1(x)}{\sqrt{n}}+\frac{\tilde g_2(x)}{n}+\cdots\right),$$

for some polynomials $\tilde g_1(x)$ and $\tilde g_1(x)$, where (μ,ξ) is given in (3.12). Finer asymptotic expansions for a_n can be derived by refining the same calculations, albeit with more messy details. □

Remark 3. *Let $\varphi(z)=\varphi_1 z+\varphi_2 z^2+\cdots$ be a formal power series with $\varphi_1\neq 0$. Then*

$$[z^n]\sum_{k\geqslant 0}d(\varphi(z))^{k+\omega_0}\prod_{1\leqslant j\leqslant k}\big(e(\varphi(z))^{j+\omega}-1\big)^{\alpha}\simeq c'\rho^n n^{n+\alpha(\frac12+\omega)},$$

where $c'=e^{\frac{\alpha\varphi_2\pi^2}{12e_1\varphi_1^2}}c$, c being given in (5.6), and $\rho=\frac{12e_1\varphi_1}{e\alpha\pi^2}$. In particular, when $\varphi(z)=\frac{z}{1-z}$, ρ remains unchanged and only the leading constant is different: $c'=e^{\frac{\alpha\pi^2}{12e_1}}c$.

6. Applications I. Univariate asymptotics

We group in this section various examples (mostly from the OEIS) according to the pair (α,ω).

6.1. Λ-row-Fishburn matrices and examples with $(\alpha,\omega)=(1,0)$.

We derive a general asymptotic approximation to the number of Λ-row-Fishburn matrices and discuss some other examples.

6.1.1. Λ-*row-Fishburn matrices.* From Theorem 21, it is clear that no matter how widely we choose the nonnegative integers as entries, the number of the resulting row-Fishburn matrices of size n depends only on the numbers of 1s and 2s as far as the leading order asymptotics is concerned, provided that the generating function satisfies (6.1).

Corollary 22. *Let Λ be a multiset of nonnegative integers with the generating function*

$$\Lambda(z) = 1 + \sum_{\lambda\in\Lambda} z^{\lambda} = 1 + \lambda_1 z + \lambda_2 z^2 + \cdots . \tag{6.1}$$

If $\Lambda'(0) = \lambda_1 > 0$, then the number of Λ-row-Fishburn matrices of size n satisfies

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left(\Lambda(z)^j - 1\right) \simeq c\rho^n n^{n+\frac{1}{2}} \quad \textit{with } (c,\rho) = \left(\frac{12}{\pi^{3/2}}\, e^{\frac{\pi^2}{12}\left(\frac{\lambda_2}{\lambda_1^2}-\frac{1}{2}\right)}, \frac{12\lambda_1}{e\pi^2}\right). \tag{6.2}$$

Proof. By Theorem 21 with $(d(z), e(z)) = (1, \Lambda(z))$. □

In particular, this corollary applies to the following OEIS sequences.

OEIS	Λ	$\Lambda(z)$	(λ_1,λ_2)	c	ρ
A179525	$\{0,1\}$	$1+z$	$(1,0)$	$\frac{12}{\pi^{3/2}}\, e^{-\frac{\pi^2}{24}}$	$\frac{12}{e\pi^2}$
A289316	$\{0\}\cup\{2k-1 : k\in\mathbb{Z}^+\}$	$\frac{1+z-z^2}{1-z^2}$	$(1,0)$	$\frac{12}{\pi^{3/2}}\, e^{-\frac{\pi^2}{24}}$	$\frac{12}{e\pi^2}$
A207433	$\{0,1,2\}$	$\frac{1-z^3}{1-z}$	$(1,1)$	$\frac{12}{\pi^{3/2}}\, e^{\frac{\pi^2}{24}}$	$\frac{12}{e\pi^2}$
A158691	$\mathbb{Z}_{\geqslant 0}$	$\frac{1}{1-z}$	$(1,1)$	$\frac{12}{\pi^{3/2}}\, e^{\frac{\pi^2}{24}}$	$\frac{12}{e\pi^2}$
A289313	$\{0,1,1,2,2,\ldots\}$	$\frac{1+z}{1-z}$	$(2,2)$	$\frac{12}{\pi^{3/2}}$	$\frac{24}{e\pi^2}$

Table 3. The large-n asymptotics (6.2) of some OEIS sequences that correspond to the enumeration of Λ-row-Fishburn matrices with different Λ. Here we split the pair (c,ρ) for clarity and group the sequences with the same pair (λ_1,λ_2).

The sequence A289313 can also be interpreted as the number of upper triangular matrices with integer entries (positive and negative) whose sum of absolute entries is n, and no row sums (in absolute entries) to zero.

As another example, let Λ be the set of prime numbers including 1. Then since $\Lambda(z) = 1 + z + z^2 + \cdots$, we see immediately from Corollary 22 that the number of row-Fishburn matrices with prime numbers and 1 as entries satisfies

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left(\Big(1 + z + \sum_{p \text{ prime}} z^p\Big)^j - 1\right) \simeq c\rho^n n^{n+\frac{1}{2}} \quad \text{with } (c,\rho) = \left(\tfrac{12}{\pi^{3/2}}\, e^{\frac{\pi^2}{24}}, \tfrac{12}{e\pi^2}\right),$$

which has the same leading asymptotics as A207433 and A158691.

Even in the case when each i appears $i!$ times for $i = 1,\ldots,n$, we still have

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left(\Big(\sum_{i\geqslant 0} i!z^i\Big)^j - 1\right) \simeq c\rho^n n^{n+\frac{1}{2}} \quad \text{with } (c,\rho) = \left(\tfrac{12}{\pi^{3/2}}\, e^{\frac{\pi^2}{8}}, \tfrac{12}{e\pi^2}\right).$$

6.1.2. *Some OEIS sequences.* Some other OEIS examples with $(\alpha,\omega) = (1,0)$ are compiled in the following table, where they all satisfy the asymptotic pattern

$$[z^n]\sum_{k\geqslant 0} d(z)^k \prod_{1\leqslant j\leqslant k}\big(e(z)^j - 1\big) \simeq c\rho^n n^{n+\frac{1}{2}}. \tag{6.3}$$

OEIS	$d(z)$	$e(z)$	(d_1, e_1, e_2)	(c, ρ)
A158690	1	e^z	$(0, 1, \frac{1}{2})$	$\big(\frac{12}{\pi^{3/2}}, \frac{12}{e\pi^2}\big)$
A196194	$\frac{z}{e^z-1}$	e^z	$\big(-\frac{1}{2}, 1, \frac{1}{2}\big)$	$\big(\frac{6\sqrt{2}}{\pi^{3/2}}, \frac{12}{e\pi^2}\big)$
A207214	e^z	e^z	$(1, 1, \frac{1}{2})$	$\big(\frac{24}{\pi^{3/2}}, \frac{12}{e\pi^2}\big)$
A207386	1	$\frac{1+z}{1+z^3}$	$(0, 1, 0)$	$\big(\frac{12}{\pi^{3/2}} e^{-\frac{\pi^2}{24}}, \frac{12}{e\pi^2}\big)$
A207397	1	$\frac{1+z}{1+z^2}$	$(0, 1, -1)$	$\big(\frac{12}{\pi^{3/2}} e^{-\frac{\pi^2}{8}}, \frac{12}{e\pi^2}\big)$
A207556	$1+z$	$1+z$	$(1, 1, 0)$	$\big(\frac{24}{\pi^{3/2}} e^{-\frac{\pi^2}{24}}, \frac{12}{e\pi^2}\big)$

Table 4. Some OEIS examples with $(\alpha,\omega) = (1,0)$*; they all satisfy the asymptotic pattern* (6.3) *with* (c,ρ) *given in the last column. All* ρ*'s are the same because* $e'(0) = 1$.

Note that the Taylor expansions of $e(z)$ in the two cases A207386 and A207397 both contain negative coefficients. As an extreme example, we consider $d(z) = 1 - \frac{z}{1-z}$ and $e(z) = 1 - \frac{z}{1-z}$, so that all coefficients of $d(z)$ and $e(z)$ are negative except the constant terms. Then we still have

$$[z^n]\sum_{k\geqslant 0}\Big(\frac{1-2z}{1-z}\Big)^k \prod_{1\leqslant j\leqslant k}\bigg(\Big(\frac{1-2z}{1-z}\Big)^j - 1\bigg) \simeq c\rho^n n^{n+\frac{1}{2}}, \quad \text{with} \quad (c,\rho) = \big(\tfrac{24}{\pi^{3/2}} e^{-\frac{\pi^2}{8}}, -\tfrac{12}{e\pi^2}\big),$$

with a negative ρ or an alternating sequence.

6.1.3. *Minor variants.* Consider A207652 whose generating function does not have the same pattern (5.1); yet this sequence has the same leading order asymptotics as A179525 (see Table 3):

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\frac{(1+z)^j - 1}{1-z^j} \simeq c\rho^n n^{n+\frac{1}{2}}, \quad \text{with} \quad (c,\rho) = \big(\tfrac{12}{\pi^{3/2}} e^{-\frac{\pi^2}{24}}, \tfrac{12}{e\pi^2}\big).$$

This is because the extra product

$$\prod_{1\leqslant j\leqslant k}\frac{1}{1-z^j} = 1 + z + O(|z|^2) \tag{6.4}$$

is asymptotically negligible when $z \asymp n^{-1}$.

Similarly, the sequence A207653 satisfies

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\frac{1-(1-z)^{2j-1}}{1-z^{2j-1}} \simeq c\rho^n n^{n+\frac{1}{2}}, \quad \text{with} \quad (c,\rho) = \big(\tfrac{12}{\pi^{3/2}} e^{\frac{\pi^2}{24}}, \tfrac{12}{e\pi^2}\big). \tag{6.5}$$

which has the same leading-order asymptotics as A158691 because of

$$\prod_{1\leqslant j\leqslant k} \frac{1-(1-z)^{2j-1}}{1-z^{2j-1}} = \prod_{1\leqslant j\leqslant k} \left(1-(1-z)^{2j-1}\right) \times \left(1+O\left(|z|\right)\right),$$

when $z \asymp n^{-1}$, and the identity

$$\sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \left(1-(1-z)^{2j-1}\right) = \sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \left((1-z)^{-j}-1\right).$$

Another example is A207434, which is defined by

$$b_n := n[z^n] \log\left(\sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \left((1+z)^j-1\right)\right).$$

This is not of our format (5.1) but the leading asymptotics can be quickly linked to that of A179525, the number of primitive row-Fishburn matrices; see (1.5). Let $a_n := [z^n] \sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \left((1+z)^j-1\right)$. By the relation

$$b_n = na_n - \sum_{1\leqslant j<n} b_j a_{n-j} \qquad (n \geqslant 1),$$

and the factorial growth of the coefficients (1.5), we then deduce that

$$b_n \simeq na_n \simeq c\rho^n n^{n+\frac{3}{2}}, \text{ with } (c,\rho) = \left(\tfrac{12}{\pi^{3/2}}\, e^{-\frac{\pi^2}{24}}, \tfrac{12}{e\pi^2}\right).$$

6.1.4. *Recursive variants.* Consider first the sequence A186737 whose generating function is defined recursively by

$$f(z) = \sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \left((1+zf(z))^j-1\right) = 1+z+3z^2+14z^3+82z^4+563z^5+\cdots.$$

This is close to the framework (5.1) but Theorem 21 fails because f is not only recursive. However, if we apply naively Theorem 21, then we obtain the correct asymptotic approximation

$$[z^n]\sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} ((1+zf(z))^j-1) \simeq c\rho^n n^{n+\frac{1}{2}}, \text{ with } (c,\rho) = \left(\tfrac{12}{\pi^{3/2}} e^{\frac{\pi^2}{24}}, \tfrac{12}{e\pi^2}\right),$$

consistent with the expression derived by Kotěšovec on the OEIS page.

While Theorem 21 does not apply, the proof there does. More precisely, we truncate first all terms in the Taylor expansion of f with powers $k > n$, so that the resulting series becomes a polynomial, and then perform the change of variables $e^y = f(z)$. Then the local expansion of the solution is given by

$$z = y - \tfrac{5}{2}y^2 - \tfrac{7}{6}y^3 - \tfrac{65}{24}y^4 + \cdots,$$

and the remaining analysis follows the same procedure as the proof of Theorem 21.

Similarly, the sequence A224885 is defined by the generating function

$$f(z) = 1+z+\sum_{k\geqslant 2} \prod_{1\leqslant j\leqslant k} \left(f(z)^j-1\right).$$

Then $f(z) = 1+z+2z^2+15z^3+143z^4+1552z^5+\cdots$, and we deduce that

$$[z^n]f(z) \simeq c\rho^n n^{n+\frac{1}{2}}, \text{ with } (c,\rho) = \left(\tfrac{12}{\pi^{3/2}}\, e^{\frac{\pi^2}{8}}, \tfrac{12}{e\pi^2}\right).$$

6.2. **Λ-Fishburn matrices and examples with $(\alpha, \omega) = (2, 0)$.** We now consider the case when $(\alpha, \omega) = (2, 0)$, beginning with the asymptotics of Λ-Fishburn matrices.

6.2.1. *Λ-Fishburn matrices.*

Corollary 23. *Let Λ be a multiset of nonnegative integers with the generating function $\Lambda(z)$ defined as in (6.1). If $\lambda_1 > 0$, then the number of Fishburn matrices of size n satisfies*

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left(1-\Lambda(z)^{-j}\right) \simeq c\rho^n n^{n+1} \quad \textit{with } (c,\rho) = \left(\frac{12\sqrt{6}}{\pi^2}e^{\frac{\pi^2}{6}\left(\frac{\lambda_2}{\lambda_1^2}-\frac{1}{2}\right)}, \frac{6\lambda_1}{e\pi^2}\right).$$

Proof. By (2.5) and then Theorem 21 with $d(z) = e(z) = \Lambda(z)$ and $\alpha = 2$. $\square$

A few OEIS examples to which this corollary applies are collected in Table 5.

OEIS	Λ	$\Lambda(z)$	(λ_1, λ_2)	(c, ρ)
A022493	$\mathbb{Z}_{\geqslant 0}$	$\frac{1}{1-z}$	$(1, 1)$	$\left(\frac{12\sqrt{6}}{\pi^2}e^{\frac{\pi^2}{12}}, \frac{6}{e\pi^2}\right)$
A138265	$\{0, 1\}$	$1+z$	$(1, 0)$	$\left(\frac{12\sqrt{6}}{\pi^2}e^{-\frac{\pi^2}{12}}, \frac{6}{e\pi^2}\right)$
A289317	$\{0\} \cup \{2k-1 : k \in \mathbb{Z}^+\}$	$\frac{1+z-z^2}{1-z^2}$	$(1, 0)$	$\left(\frac{12\sqrt{6}}{\pi^2}e^{-\frac{\pi^2}{12}}, \frac{6}{e\pi^2}\right)$
A289312	$\{0\} \cup 2\mathbb{Z}^+$	$\frac{1+z}{1-z}$	$(2, 2)$	$\left(\frac{12\sqrt{6}}{\pi^2}, \frac{12}{e\pi^2}\right)$

Table 5. The large-n asymptotics (of the form $c\rho^n n^{n+1}$) of some OEIS sequences that correspond to the enumeration of Λ-Fishburn matrices with different Λ.

In particular, we see from Table 5 that Zagier's result (1.1) for the asymptotics of Fishburn numbers corresponds to A022493. Also the result for A138265 improves the crude bound given in [32]; see also [7].

Written differently, Corollary 23 also implies the asymptotic relation

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left(1-\Big(1+z+\sum_{p:\,\mathrm{prime}} z^p\Big)^{-j}\right) \simeq [z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left(1-(1-z)^j\right).$$

Similarly, the same asymptotic approximation holds in the case when $\Lambda = \{0, 1, \ldots, m-1\}$ (studied in [13]) with the generating function $\Lambda(z) = 1 + z + \cdots + z^{m-1}$, $m \geqslant 3$.

On the other hand, we also have for the so-called r-Fishburn numbers [22] with $\Lambda(z) = (1-z)^{-r}$

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left(1-(1-z)^{rj}\right) \simeq c\rho^n n^{n+1} \quad \text{with } (c,\rho) = \left(\frac{12\sqrt{6}}{\pi^2}e^{\frac{\pi^2}{12r}}, \frac{6r}{e\pi^2}\right).$$

This asymptotic estimate holds for any $r > 0$ (not necessarily integers). Furthermore,

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left(1-\Big(\sum_{i\geqslant 0} i!z^i\Big)^{-j}\right) \simeq c\rho^n n^{n+1} \quad \text{with } (c,\rho) = \left(\frac{12\sqrt{6}}{\pi^2}e^{\frac{\pi^2}{4}}, \frac{6}{e\pi^2}\right).$$

6.2.2. *Other OEIS examples.* We discuss three other OEIS sequences with $(\alpha, \omega) = (2, 0)$. Consider first A079144, which enumerates labelled interval orders on n points [7] with $d(z) = e(z) = e^z$, and we obtain

$$
\begin{aligned}
[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left(1-e^{-jz}\right) &= [z^n]\sum_{k\geqslant 0} e^{(k+1)z}\prod_{1\leqslant j\leqslant k}\left(e^{jz}-1\right)^2 \\
&\simeq c\rho^n n^{n+1},\ \text{ with } (c,\rho) = \left(\tfrac{12\sqrt{6}}{\pi^2}, \tfrac{6}{e\pi^2}\right).
\end{aligned}
$$

Alternatively, (1.2) provides an alternative proof for this asymptotic estimate and a finer expansion; see [44].

Consider now A207651, the generating function of this sequence is different from A022493, the Fishburn numbers, but they satisfy the same asymptotic relation (see (1.1))

$$
[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\frac{1-(1-z)^j}{1-z^j} \simeq c\rho^n n^{n+1},\ \text{ with } (c,\rho) = \left(\tfrac{12\sqrt{6}}{\pi^2}e^{\frac{\pi^2}{12}}, \tfrac{6}{e\pi^2}\right),
$$

since the additional product is again asymptotically negligible; see (6.4).

The last sequence is A035378:

$$
[z^n]\sum_{k\geqslant 1}\prod_{1\leqslant j\leqslant k}\left(1-(z-1)^j\right) = [z^n]\sum_{k\geqslant 0}(z-1)^{-k-1}\prod_{1\leqslant j\leqslant k}\left(1-(z-1)^{-j}\right)^2.
$$

Theorem 21 does not apply directly but our approach does by rewriting the GF as (by grouping the terms in pairs)

$$
\sum_{k\geqslant 0}\frac{1}{(1-z)^{2k+1}}\left(\frac{1}{1-z}\left(1+\frac{1}{(1-z)^{2k+1}}\right)^2-1\right)\prod_{1\leqslant j\leqslant 2k}\left(\frac{1}{(1-z)^j}-1\right)^2;
$$

we then derive the approximation

$$
[z^n]\sum_{k\geqslant 1}\prod_{1\leqslant j\leqslant k}\left(1-(z-1)^j\right) \simeq c\rho^n n^{n+1},\ \text{ with } (c,\rho) = \left(\tfrac{48\sqrt{3}}{\pi^2}\,e^{\frac{\pi^2}{48}}, \tfrac{24}{e\pi^2}\right),
$$

consistent with that provided on the OEIS webpage of A035378 by Kotěšovec; see also [44, Sec. 5].

6.3. **Examples with $\omega \neq 0$.** We gather some examples in the following table, where we use the form

$$
a_n := [z^n]\sum_{k\geqslant 0} d_k(z)\prod_{1\leqslant j\leqslant k}\left(e_j(z)-1\right),
$$

with $(d_k(z), e_j(z))$ given in the second column.

OEIS	$(d_k(z), e_j(z))$	$a_n n^{-n} \simeq$	(c, ρ)
A215066	$(1, e^{(2j-1)z})$	$c\rho^n n^n$	$\left(\frac{2\sqrt{3}}{\pi}, \frac{24}{e\pi^2}\right)$
A209832	$(e^{(k+1)z}, e^{(2j-1)z})$	$c\rho^n n^n$	$\left(\frac{2\sqrt{6}}{\pi}, \frac{24}{e\pi^2}\right)$
A214687	$\left(e^{2kz}, e^{(2j-1)z}\right)$	$c\rho^n n^n$	$\left(\frac{4\sqrt{3}}{\pi}, \frac{24}{e\pi^2}\right)$
A207569	$\left(1, (1+z)^{2j-1}\right)$	$c\rho^n n^n$	$\left(\frac{2\sqrt{3}}{\pi} e^{-\frac{\pi^2}{48}}, \frac{24}{e\pi^2}\right)$
A207570	$\left(1, (1+z)^{3j-2}\right)$	$c\rho^n n^{n-\frac{1}{6}}$	$\left(\frac{\Gamma(\frac{2}{3})3^{5/6}}{2^{1/3}\pi^{7/6}} e^{-\frac{\pi^2}{72}}, \frac{36}{e\pi^2}\right)$
A207571	$\left(1, (1+z)^{3j-1}\right)$	$c\rho^n n^{n+\frac{1}{6}}$	$\left(\frac{12^{2/3}}{\pi^{5/6}\Gamma(\frac{2}{3})} e^{-\frac{\pi^2}{72}}, \frac{36}{e\pi^2}\right)$

In general,

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left((1+z)^{pj-s}-1\right) \simeq c\rho^n n^{n+\frac{1}{2}-\frac{s}{p}},$$

for $0 < s < p$ (not necessarily integers), where

$$(c, \rho) = \left(\frac{\sqrt{\pi}}{\Gamma\left(1-\frac{s}{p}\right)}\left(\frac{\pi^2}{12}\right)^{\frac{s}{p}-1} e^{-\frac{\pi^2}{24p}}, \frac{12p}{e\pi^2}\right).$$

A minor variant of A207569 is the sequence A207654 for which we have the same asymptotic approximation

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\frac{(1+z)^{2j-1}-1}{1-z^{2j-1}} \simeq c\rho^n n^n, \quad \text{with } (c, \rho) = \left(\frac{2\sqrt{3}}{\pi} e^{-\frac{\pi^2}{48}}, \frac{24}{e\pi^2}\right),$$

because the extra product is asymptotically negligible; see also (6.5).

The last example is A207557:

$$f(z) := \sum_{k\geqslant 0}(1+z)^{-k(k-1)}\prod_{1\leqslant j\leqslant k}\left((1+z)^{2j-1}-1\right),$$

which can be transformed, by the Rogers-Fine identity[1] [15], into

$$f(z) = 1 + z^{-1}\sum_{k\geqslant 1}(1+z)^{2k+1}\prod_{1\leqslant j\leqslant k}\left((1+z)^{2j-1}-1\right)^2.$$

We can then apply Theorem 21, and obtain

$$\begin{aligned}[z^n]f(z) &= [z^{n+1}]\sum_{k\geqslant 1}(1+z)^{2k+1}\prod_{1\leqslant j\leqslant k}\left((1+z)^{2j-1}-1\right)^2 \\ &\simeq c_0\rho^{n+1}(n+1)^{n+1} \simeq c_0 e\rho\rho^n n^{n+1}, \quad \text{with } (c_0, \rho) = \left(\frac{2\sqrt{6}}{\pi} e^{-\frac{\pi^2}{24}}, \frac{12}{e\pi^2}\right).\end{aligned}$$

[1] For $|t|, |q| < 1$ and y not a negative power of x,

$$\sum_{k\geqslant 0} t^k \prod_{1\leqslant j\leqslant k}\frac{1-xq^j}{1-yq^j} = \sum_{k\geqslant 0}\frac{(1-xtq^{2k+1})t^k q^{k^2}}{1-t}\prod_{1\leqslant j\leqslant k}\frac{(1-xq^j)(y-xtq^j)}{(1-yq^j)(1-tq^j)}. \tag{6.6}$$

Thus $c_0 e\rho = \frac{24\sqrt{6}}{\pi^3} e^{-\frac{\pi^2}{24}}$, consistent with the expression derived by Kotěšovec on the OEIS page of A207557.

7. Applications II. Bivariate asymptotics (asymptotic distributions)

We derive in this section the various limit laws arising from the sizes of the first row and the diagonal, as well as the number of 1s in random Fishburn and row-Fishburn matrices, assuming that all matrices of the same size are equally likely to be selected. We begin with row-Fishburn matrices because they are technically simpler.

7.1. **Statistics on Λ-row-Fishburn matrices.** By Proposition 1, the number of Λ-row-Fishburn matrices of size n is given by (see (2.4))

$$a_n := [z^n] \sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \left(\Lambda(z)^j - 1\right),$$

where $\Lambda(z)$ is the generating function of the multiset Λ; see (2.3). The asymptotics of a_n is already examined in Corollary 22.

Recall that the probability generating function of a Poisson distribution with mean $\tau > 0$ is given by $e^{\tau(v-1)}$, while that of a zero-truncated Poisson (ZTP) distribution with parameter τ by

$$\frac{e^{\tau v} - 1}{e^{\tau} - 1}$$

whose mean and variance equal

$$\frac{\tau e^{\tau}}{e^{\tau} - 1} \quad \text{and} \quad \frac{\tau e^{\tau}(e^{\tau} - 1 - \tau)}{(e^{\tau} - 1)^2},$$

respectively. When $\tau = \log 2$, these become $2\log 2$ and $2(\log 2)(1 - \log 2)$, respectively. Also $\mathscr{N}(0,1)$ denotes the standard normal distribution. The notation $X_n \xrightarrow{d} X$ means convergence in distribution.

7.1.1. *Limit theorems.*

Theorem 24. *Assume $\lambda_1 > 0$ and that all Λ-row-Fishburn matrices of size n are equally likely to be selected. Then in a random matrix,*

(i) the size X_n of the first row is distributed asymptotically as zero-truncated Poisson with parameter $\log 2$:

$$X_n \xrightarrow{d} \mathrm{ZTP}(\log 2),$$

(ii) the size Y_n of the diagonal is asymptotically normally distributed with mean and variance both asymptotic to $\log n$,

$$\frac{Y_n - \log n}{\sqrt{\log n}} \xrightarrow{d} \mathscr{N}(0,1), \tag{7.1}$$

and

(iii) for the number of 1*s* Z_n*, if* $\lambda_2 > 0$*, then*

$$\frac{n - Z_n}{2} \xrightarrow{d} \mathrm{Poisson}(\tau) \quad \textit{with} \quad \tau = \tfrac{\lambda_2 \pi^2}{12\lambda_1^2}, \tag{7.2}$$

and $\mathbb{P}(Z_n = n) \to 1$ *if* $\lambda_2 = 0$.

For the diagonal size, we can also express the asymptotic distribution as $Y_n \sim \mathrm{Poisson}(\log n)$, which implies (7.1). Finer approximations are given in (7.4) and (7.5).

Proof. (i) For the first row size X_n, we begin with the generating function (see (2.7))

$$f_X(z, v) := \sum_{k \geqslant 0} \big(\Lambda(vz)^{k+1} - 1\big) \prod_{1 \leqslant j \leqslant k} \big(\Lambda(z)^j - 1\big).$$

By applying (5.5) to $(d(z), e(z)) = (\Lambda(vz), \Lambda(z))$ and to $(d(z), e(z)) = (1, \Lambda(z))$, we deduce that

$$\mathbb{E}\big(v^{X_n}\big) = \frac{[z^n] f_X(z, v)}{a_n} \simeq 2^v - 1,$$

for $v = O(1)$. This asymptotic estimate holds *a priori* pointwise for each finite v, but the same proof there gives indeed the uniformity of the error term in v when $v = O(1)$. This implies the convergence in distribution to the zero-truncated Poisson (ZTP) law with parameter $\log 2$.

(ii) Consider now the generating polynomial for the diagonal size Y_n

$$[z^n] f_Y(z, v) := [z^n] \sum_{k \geqslant 1} \prod_{1 \leqslant j \leqslant k} \big(\Lambda(vz)\Lambda(z)^{j-1} - 1\big).$$

The generating function is not of the form (5.1), but observe that

$$\Lambda(vz) = \Lambda(z)^v \big(1 + O(|z|^2)\big),$$

when $|z|$ is small. Then, when $k \asymp n$ and $|z| \asymp n^{-1}$ (taking logarithm and estimating the sum of errors), we have

$$\prod_{1 \leqslant j \leqslant k} \big(\Lambda(vz)\Lambda(z)^{j-1} - 1\big) = \Bigg(\prod_{1 \leqslant j \leqslant k} \big(\Lambda(z)^{j+v-1} - 1\big)\Bigg)\big(1 + O\big(|z| \log k\big)\big), \tag{7.3}$$

and we are in a position to apply Theorem 21, giving

$$[z^n] f_Y(z, v) = c(v) \rho^n n^{n+v-\frac{1}{2}} \big(1 + O\big(n^{-1} \log n\big)\big),$$

where

$$(c(v), \rho) = \left(\frac{\sqrt{\pi}}{\Gamma(v)} \Big(\frac{12\lambda_1}{\pi^2}\Big)^v e^{\frac{\pi^2}{12}\left(\frac{\lambda_2}{\lambda_1^2} - \frac{1}{2}\right)}, \frac{12\lambda_1}{e\pi^2}\right),$$

uniformly for $v = O(1)$. Accordingly, the probability generating function of Y_n satisfies

$$\mathbb{E}\big(v^{Y_n}\big) = \frac{[z^n] f(z, v)}{a_n} = \frac{1}{\Gamma(v)} \left(\frac{12}{\pi^2}\right)^{v-1} e^{(v-1)\log n} \big(1 + O\big(n^{-1} \log n\big)\big),$$

uniformly for $v = O(1)$. This is of the form of Quasi-Powers (see [20, 26]), and we then deduce the asymptotic normality of Y_n with optimal convergence rate:

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}\Big(\frac{Y_n-\log n}{\sqrt{\log n}}\leqslant x\Big)-\Phi(x)\right| = O\big((\log n)^{-\frac{1}{2}}\big), \tag{7.4}$$

together with the asymptotic approximations to the mean and the variance:

$$\begin{aligned}\mathbb{E}(Y_n) &= \log n+\gamma+\log\tfrac{12}{\pi^2}+O\big(n^{-1}\log n\big),\\ \mathbb{V}(Y_n) &= \log n+\gamma-\tfrac{\pi^2}{6}+\log\tfrac{12}{\pi^2}+O\big(n^{-1}(\log n)^2\big),\end{aligned} \tag{7.5}$$

where γ denotes the Euler-Mascheroni constant and $\Phi(x)$ denotes the distribution function of the standard normal distribution. For other types of Poisson approximation, see [28].

(iii) Applying the same proof of Theorem 21 to the generating function (2.9) for the number of 1s gives

$$[z^n]\sum_{k\geqslant 1}\prod_{1\leqslant j\leqslant k}\big((\Lambda(z)+\lambda_1(v-1)z)^j-1\big)\simeq c(v)v^n\rho^n n^{n+\frac{1}{2}},$$

uniformly for bounded v, where $(c(v),\rho)=\Big(\frac{12}{\pi^{3/2}}\,e^{\frac{\pi^2}{12}\left(\frac{\lambda_2}{\lambda_1^2v^2}-\frac{1}{2}\right)},\frac{12\lambda_1 v}{e\pi^2}\Big)$. This implies that if $\lambda_2>0$, then

$$\mathbb{E}\big(v^{\frac{1}{2}(n-Z_n)}\big)\simeq e^{\tau(v-1)},\quad \text{with}\quad \tau=\frac{\pi^2\lambda_2}{12\lambda_1^2}, \tag{7.6}$$

and we then obtain the limit Poisson distribution with parameter τ. If $\lambda_2=0$, then $\mathbb{E}\big(v^{n-Z_n}\big)\to 1$, a Dirac distribution. Furthermore, by the uniformity of (7.6) and Cauchy's integral representation, we also have (see [26])

$$\mathbb{P}(n-Z_n=2k)\to\frac{\tau^k}{k!}e^{-\tau}\qquad(k=0,1,\dots).$$

Similarly, for the number $Z_n^{[2]}$ of 2s, we use the generating function

$$\sum_{k\geqslant 1}\prod_{1\leqslant j\leqslant k}\big((\Lambda(z)+\lambda_2(v-1)z^2)^j-1\big),$$

and deduce that $\mathbb{E}\big(v^{Z_n^{[2]}}\big)\simeq e^{\tau(v-1)}$, with the same τ as in (7.2).

□

Stronger results such as local limit theorems can also be derived; see [26] for more information.

7.1.2. *Applications.* Consider first the case of primitive row-Fishburn matrices with $\Lambda=\{0,1\}$. Then by Theorem 24, we see that in a random primitive Fishburn matrix the first row size is asymptotically ZTP(log 2) distributed, the diagonal is asymptotically normal, while the number of 1 is obviously the same as the size of the matrix. In particular, the distribution of the diagonal corresponds to sequence A182319.

On the other hand, when $\Lambda(z)=\frac{1}{1-z}$, we have very similar behaviors for the sizes of the first row and the diagonal, but the number Z_n of 1s is asymptotically Poisson:

$$\mathbb{P}(n-Z_n=2k)\to\frac{\tau^k}{k!}e^{-\tau},\quad\text{with}\quad\tau=\frac{\pi^2}{12},$$

for $k = 0, 1, \ldots$.

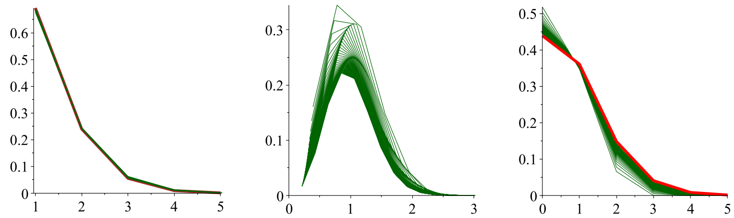


Figure 7.1. The histograms of X_n, Y_n and Z_n in the case of row-Fishburn matrices ($\Lambda(z) = \frac{1}{1-z}$) for $n = 6, \ldots, 50$: $\mathbb{P}(X_n = k)$ (left), $\mathbb{P}(Y_n = \lfloor t\mu_n \rfloor)$ (middle), and $\mathbb{P}(n - Z_n = 2k)$ (right), where $\mu_n = \mathbb{E}(Y_n)$. Then tendence to ZTP, normal and Poisson is visible in each case, as well as the corresponding convergence rate.

7.2. **Statistics on Fishburn matrices.** We consider random Λ-Fishburn matrices in this subsection. By Proposition 1, the number of Λ-Fishburn matrices of size n is given by (see (2.5))

$$a_n := [z^n] \sum_{k \geqslant 0} \prod_{1 \leqslant j \leqslant k} \left(1 - \Lambda(z)^{-j}\right),$$

and an asymptotic approximation is already derived in Corollary 23.

7.2.1. *Limit theorems.*

Theorem 25. *Assume $\lambda_1 > 0$ and that all Λ-Fishburn matrices of size n are equally likely to be selected. Then in a random matrix, the size X_n of the first row and the diagonal size Y_n are both asymptotically normally distributed with logarithmic mean and variance in the following sense*

$$\frac{X_n - \log n}{\sqrt{\log n}} \xrightarrow{d} \mathscr{N}(0,1), \quad \textit{and} \quad \frac{Y_n - 2\log n}{\sqrt{2\log n}} \xrightarrow{d} \mathscr{N}(0,1), \tag{7.7}$$

and if $\lambda_2 > 0$, then the number Z_n of 1s is asymptotically Poisson distributed

$$\frac{n - Z_n}{2} \xrightarrow{d} \text{Poisson}(\tau) \quad \textit{with} \quad \tau = \frac{\lambda_2 \pi^2}{12\lambda_1^2}, \tag{7.8}$$

otherwise, $\lambda_2 = 0$ implies that $\mathbb{P}(Z_n = n) \to 1$.

Proof. (i) We begin with the generating function (see (2.11)) for the first row size

$$f_X(z, v) := \Lambda(vz) \sum_{k \geqslant 0} \Lambda(z)^k \prod_{1 \leqslant j \leqslant k} \left(\left(\Lambda(vz)\Lambda(z)^{j-1} - 1\right)\left(\Lambda(z)^j - 1\right)\right).$$

By (7.3) and the expansion $\Lambda(vz) = 1 + O(|z|)$ for small $|z|$, we have

$$f_X(z, v) = \left(\sum_{k \geqslant 0} \Lambda(z)^k \prod_{1 \leqslant j \leqslant k} \left(\left(\Lambda(z)^{j+v-1} - 1\right)\left(\Lambda(z)^j - 1\right)\right)\right)\left(1 + O(|z| \log n)\right),$$

when $|z| \asymp n^{-1}$.

Similar to Theorem 21, we first derive, by the same methods of proof of Proposition 20, that

$$[z^n]\sum_{k\geqslant 0} e^{kz} \prod_{1\leqslant j\leqslant k}\big(e^{(j+\omega)z}-1\big)\big(e^{jz}-1\big) \simeq c_0(\omega)\rho^n n^{n+\omega+1}, \tag{7.9}$$

where

$$(c_0(\omega),\rho) = \left(\frac{2\sqrt{6}}{\Gamma(1+\omega)}\Big(\frac{6}{\pi^2}\Big)^{1+\omega}, \frac{6}{e\pi^2}\right).$$

[Briefly, α is almost 2 in the proof of Proposition 20, and the largest terms occur when $k\sim\mu n$ and $n|z|\sim\xi$ with (μ,ξ) as in (3.12), so that e^{kz} contributes an extra factor 2.]

We now make the change of variables $\Lambda(z)=e^y$, and follow the same proof procedure of Theorem 21, yielding

$$[z^n]\sum_{k\geqslant 0}\Lambda(z)^k \prod_{1\leqslant j\leqslant k}\Big(\big(\Lambda(z)^{j+v-1}-1\big)\big(\Lambda(z)^j-1\big)\Big) \simeq c(v)\rho^n n^{n+v},$$

where

$$(c(v),\rho) = \left(\frac{2\sqrt{6}}{\Gamma(v)}\Big(\frac{6}{\pi^2}\Big)^{v} e^{\frac{\pi^2}{12}\left(\frac{\lambda_2}{\lambda_1^2}-\frac{1}{2}\right)}, \frac{6\lambda_1}{e\pi^2}\right).$$

We then deduce that

$$\mathbb{E}\big(v^{X_n}\big) = \frac{1}{\Gamma(v)}\Big(\frac{6}{\pi^2}\Big)^{v-1} n^{v-1}\big(1+O\big(n^{-1}\log n\big)\big),$$

uniformly for $v=O(1)$, and the asymptotic normality of X_n then follows again from the Quasi-Powers theorem [20, 26] or a standard characteristic function argument. Finer results such as (7.4) and (7.5) can also be derived.

(ii) For the size of the diagonal Y_n, we now have the generating function (see (2.12))

$$f_Y(z,v) := \Lambda(vz)\sum_{k\geqslant 0}\Lambda(z)^k \prod_{1\leqslant j\leqslant k}\big(\Lambda(vz)\Lambda(z)^{j-1}-1\big)^2.$$

By (7.3), the same arguments used in (i) for X_n and Theorem 21, we deduce that

$$[z^n]f_Y(z,v) = c(v)\rho^n n^{n+2v-1}\big(1+O\big(n^{-1}\log n\big)\big),$$

where

$$(c(v),\rho) = \left(\frac{2\sqrt{6}}{\Gamma(v)^2}\Big(\frac{6}{\pi^2}\Big)^{2v-1} e^{\frac{\pi^2}{12}\left(\frac{\lambda_2}{\lambda_1^2}-\frac{1}{2}\right)}, \frac{6\lambda_1}{e\pi^2}\right).$$

It follows that

$$\mathbb{E}\big(v^{Y_n}\big) = \frac{1}{\Gamma(v)^2}\Big(\frac{6}{\pi^2}\Big)^{2(v-1)} n^{2(v-1)}\big(1+O\big(n^{-1}\log n\big)\big),$$

uniformly for $v=O(1)$. The asymptotic normality then follows from Quasi-Powers Theorem.

(iii) Since $\lambda_1>0$, we can apply Theorem 21 to the generating function (2.13) for the number Z_n of 1s, which is

$$f_Z(z,v) := \sum_{k\geqslant 0}(\Lambda(z)+\lambda_1(v-1)z)^{k+1}\prod_{1\leqslant j\leqslant k}\big((\Lambda(z)+\lambda_1(v-1)z)^j-1\big)^2,$$

and we deduce that

$$\mathbb{E}\big(v^{\frac{1}{2}(n-Z_n)}\big) \simeq e^{\tau(v-1)}, \quad \text{with} \quad \tau = \tfrac{\pi\lambda_2}{6\lambda_1^2},$$

which leads to a degenerate limit law when $\lambda_2 = 0$ and a Poisson limit law otherwise. The number of 2s follows the same law.

□

The most widely studied parameter is the size X_n of the first row when $\Lambda(z) = \frac{1}{1-z}$, the Fishburn matrices. It appeared in Stoimenow's study [39] on chord diagrams, and later examined by Zagier in [44]. Then the limiting distribution of X_n was raised as an open question in [8, 30]. The generating function $f_X(z,v)$ for the first row size has been derived in several papers; see, for example, [4, 6, 21, 30, 43], A175579 and Section 2 for several other quantities with the same distribution as X_n. See also Table 6 and Figure 7.2 for the distribution of small n and graphical renderings.

$n\backslash k$	1	2	3	4	5	6	7
1	1						
2	1	1					
3	2	2	1				
4	5	6	3	1			
5	15	21	12	4	1		
6	53	84	54	20	5	1	
7	217	380	270	110	30	6	1

$n\backslash k$	1	2	3	4	5	6	7
1	1						
2	0	2					
3	0	1	4				
4	0	2	5	8			
5	0	5	14	18	16		
6	0	15	47	67	56	32	
7	0	53	183	287	267	160	64

Table 6. The number of Fishburn matrices of size n with first row size equal to k (left) and the diagonal size to k (right) for $n = 1,\dots,7$. The table on the left corresponds to A175579.

The mean and the variance of X_n satisfy

$$\begin{aligned}
\mathbb{E}(X_n) &= \log n + \gamma - \log\tfrac{\pi^2}{6} + O\big(n^{-1}\log n\big),\\
\mathbb{V}(X_n) &= \log n + \gamma - \tfrac{\pi^2}{6} - \log\tfrac{\pi^2}{6} + O\big(n^{-1}(\log n)^2\big).
\end{aligned}$$

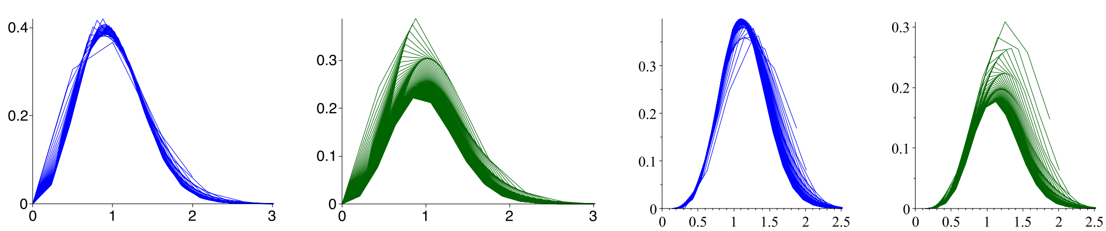


Figure 7.2. The histograms of X_n and Y_n (Fishburn matrices) for $n = 6,\dots,100$: $\sigma_n(X)\mathbb{P}(X_n = \lfloor t\mu_n(X)\rfloor)$ (first), $\mathbb{P}(X_n = \lfloor t\mu_n(X)\rfloor)$ (second), $\sigma_n(Y)\mathbb{P}(Y_n = \lfloor t\mu_n(Y)\rfloor)$ (third), $\mathbb{P}(Y_n = \lfloor t\mu_n(Y)\rfloor)$ (fourth), where $\mu_n(W)$ and $\sigma_n^2(W)$ denote the corresponding mean and variance of W_n, respectively.

8. A FRAMEWORK FOR MATRICES WITHOUT 1S AND SELF-DUAL MATRICES

We discuss in this section the extension to the situation when $e_1 = 0$ and $e_2 \neq 00$ of the general framework (5.1) we examined in Section 5. The general asymptotic expressions (5.5) and (5.6) certainly fail in such a case as the leading constant involves e_1 in the denominator.

In addition to providing a better understanding of Fishburn matrices in more general situations, our consideration of (5.1) with $e_1 = 0$ and $e_2 > 0$ was also motivated by asymptotic enumeration of the self-dual Fishburn matrices, another conjecture raised in Jelínek [30]. In particular, the two asymptotic approximations (1.10) and (1.11) will follow readily from our general result Theorem 26 or Corollary 29. Furthermore, as in Sections 6 and 7, our framework will be equally useful in characterizing the asymptotic distributions of diverse statistics in random Fishburn matrices, which we briefly explore in this section.

8.1. Asymptotics of (5.1) with $e_1 = 0$ and $e_2 > 0$.

Theorem 26. *Assume $\alpha \in \mathbb{Z}^+$ and $\omega_0, \omega \in \mathbb{C}$. Given any two formal power series $e(z) = 1 + \sum_{j\geqslant 1} e_j z^j$ and $d(z) = 1 + \sum_{j\geqslant 1} d_j z^j$ satisfying $e_1 = 0$, $e_2 > 0$, and*

$$\alpha e_3 \pi^2 + 12 d_1 e_2 \log 2 > 0, \tag{8.1}$$

we have

$$[z^n] \sum_{k\geqslant 0} d(z)^{k+\omega_0} \prod_{1\leqslant j\leqslant k} \left(e(z)^{j+\omega} - 1\right)^{\alpha} = c e^{\beta\sqrt{n}} \rho^{\frac{1}{2}n} n^{\frac{1}{2}(n+\alpha)+\alpha\omega} \left(1 + O\left(n^{-\frac{1}{2}}\right)\right), \tag{8.2}$$

the O-term holding uniformly for bounded ω_0 and ω, where $\beta := \frac{\sqrt{6} d_1 \log 2}{\sqrt{e_2 \alpha}\,\pi} + \frac{\sqrt{\alpha}\, e_3 \pi}{2\sqrt{6}\, e_2^{3/2}}$, $\rho := \frac{6 e_2}{e\pi^2 \alpha}$, and

$$c := \frac{\sqrt{3}}{\sqrt{2}\,\alpha\pi} \left(\frac{1}{\Gamma(1+\omega)} \sqrt{\frac{12}{\alpha\pi}} \left(\frac{6}{\alpha\pi^2}\right)^{\omega} \right)^{\alpha} 2^{-\frac{d_1^2}{2e_2} - \frac{3d_1 e_3}{4e_2^2} + \frac{d_2}{e_2}} e^{-\frac{d_1^2}{4\alpha e_2} - \frac{\alpha\pi^2}{12}\left(\frac{7e_3^2}{8e_2^3} - \frac{e_4}{e_2^2} + \frac{1}{2}\right) + \frac{3d_1^2}{2e_2\alpha\pi^2}(\log 2)^2}.$$

Note that $\beta > 0$ is equivalent to the condition (8.1). When $\beta = 0$ (and $e_2 > 0$), asymptotic periodicities emerge (depending on the parity of n) and complicate the corresponding expressions. Instead of formulating a general heavy result, we will be content with ourselves with the consideration of Fishburn matrices with $\lambda_1 = \lambda_3 = \cdots = \lambda_{2m-1} = 0$ but $\lambda_2, \lambda_{2m+1} > 0$ in Section 8.4.

Following the same principle we used above, we consider first the corresponding exponential version, and then prove the theorem by a change of variables argument; see Sections 4.2 and 5.

Proposition 27. *For large n, $\alpha \in \mathbb{Z}^+$, and $\omega \in \mathbb{C}$,*

$$[z^n] \sum_{k\geqslant 0} e^{kz} \prod_{1\leqslant j\leqslant k} \left(e^{(j+\omega)z^2} - 1\right)^{\alpha} = c e^{\beta\sqrt{n}} \rho^{\frac{1}{2}n} n^{\frac{1}{2}(n+\alpha)+\alpha\omega} \left(1 + O\left(n^{-\frac{1}{2}}\right)\right), \tag{8.3}$$

the O-term holding uniformly for bounded ω, where $\beta = \frac{\sqrt{6}\log 2}{\sqrt{\alpha}\pi}$, and

$$(c, \rho) = \left(\frac{\sqrt{3}}{\sqrt{2}\,\alpha\pi} \left(\frac{1}{\Gamma(1+\omega)} \sqrt{\frac{12}{\alpha\pi}} \left(\frac{6}{\alpha\pi^2}\right)^{\omega} \right)^{\alpha} e^{-\frac{1}{4\alpha} + \frac{3}{2\alpha\pi^2}(\log 2)^2}, \frac{6}{e\pi^2\alpha} \right).$$

Proof. Let $a_{n,k} := [z^n] e^{kz} A_k\left(z^2\right)^{\alpha}$ with $A_k(z) := \prod_{1\leqslant j\leqslant k}\left(e^{jz} - 1\right)$. Since the proof follows closely that of Theorem 5 and of Proposition 20, we sketch the major differences.

First of all, we use the simple upper bounds

$$a_{n,k} \leqslant r_0^{-n} e^{kr_0} A_k\big(r_0^2\big)^\alpha = r^{-\frac{1}{2}n} e^{k\sqrt{r}} A_k(r)^\alpha,$$

with r choosing to be the saddle-point of $r^{-\frac{1}{2}n} A_k(r)^\alpha$ and $r = r_0^2$, so then all estimates, bounds and arguments used in Section 3 for $A_k(r)$ can be directly applied with n there replaced by $\frac{n}{2}$. Thus the sum of $a_{n,k}$ over $1 \leqslant k \leqslant \frac{1}{2\alpha}k_-$ and $k > \frac{1}{2\alpha}k_+$ is asymptotically negligible, where $k_\pm = \mu n \pm \sqrt{2}\sigma n^{\frac{5}{8}}$ being the same as in (3.18) with $\mu = \frac{12}{\pi^2}\log 2$ and σ the same as in (3.13).

Then, in the range $\frac{1}{2\alpha}k_- \leqslant k \leqslant \frac{1}{2\alpha}k_+$, most contribution to the corresponding Cauchy integral comes from a small neighborhood of $|\theta| \leqslant \theta_0$ when $z = re^{i\theta}$ and $\theta_0 := 6n^{-\frac{3}{8}}$ as in (3.22), because

$$a_{n,k} = \frac{1}{2\pi i}\left(\int_{\substack{|z|=r_0\\|\theta|\leqslant\theta_0}} + \int_{\substack{|z|=r_0\\ \theta_0<|\theta|\leqslant\pi}}\right) z^{-n-1} e^{kz} A_k\big(z^2\big)^\alpha \,\mathrm{d}z \qquad \big(r_0 \asymp n^{-\frac{1}{2}}\big),$$

where, by (3.20), the second integral is bounded from above by

$$\begin{aligned} &O\bigg(r_0^{-n} e^{kr_0} A_k\big(r_0^2\big)^\alpha \int_{\theta_0}^{\frac{1}{2}\pi} e^{-\frac{\alpha}{2\pi^2}k^2 r_0^2\theta^2}\,\mathrm{d}\theta + r_0^{-n} A_k\big(r_0^2\big)^\alpha \int_{\frac{1}{2}\pi}^{\pi} e^{kr_0\cos\theta}\,\mathrm{d}\theta\bigg) \\ &\quad = O\Big(r_0^{-n} e^{kr_0} A_k\big(r_0^2\big)^\alpha e^{-c'q^2 n^{\frac{1}{5}}} + r_0^{-n} A_k\big(r_0^2\big)^\alpha\Big), \end{aligned}$$

which is negligible when compared with $n^{-K} r_0^{-n} e^{k\sqrt{r_0}} A_k\big(r_0^2\big)^\alpha$ for any $K > 0$.

Now when $\frac{1}{2\alpha}k_- \leqslant k \leqslant \frac{1}{2\alpha}k_+$ and $|\theta| \leqslant \theta_0$, we first carry out the change of variables $z^2 \mapsto w$, giving

$$J := \frac{1}{2\pi i}\int_{\substack{|z|=r_0\\|\theta|\leqslant\theta_0}} z^{-n-1} e^{kz} A_k\big(z^2\big)^\alpha \,\mathrm{d}z = \frac{1}{4\pi i}\int_{\substack{|z|=r\\|\theta|\leqslant\frac{1}{2}\theta_0}} w^{-\frac{1}{2}n-1} e^{k\sqrt{w}} A_k(w)^\alpha \,\mathrm{d}w.$$

Since the integral is, up to the factor $\frac{1}{2}e^{k\sqrt{w}}$ and the difference $\frac{1}{2}n$, of the same form as that studied in Section 3.5, a simple strategy is to compute first the asymptotics of the integral

$$J_1 := \frac{1}{4\pi i}\int_{\substack{|z|=r\\|\theta|\leqslant\frac{1}{2}\theta_0}} w^{-n-1} e^{k\sqrt{w}} A_k(w)^\alpha \,\mathrm{d}w,$$

with $\frac{1}{\alpha}k_- \leqslant k \leqslant \frac{1}{\alpha}k_+$, and then substitute $n \mapsto \frac{1}{2}n$. Then with the expansions

$$k = \frac{1}{\alpha}\Big(\frac{12}{\pi^2}\log 2 + \frac{\sigma x}{\sqrt{n}}\Big), \quad \text{and} \quad r = \frac{\alpha}{n}\Big(\frac{\pi^2}{12} + \sum_{j\geqslant 1}\frac{\tilde{\xi}_j(x)}{n^{j/2}}\Big),$$

where the expressions of the $\tilde{\xi}_j(x)$'s are too messy to be listed here (but straightforward with the aid of symbolic computation softwares), we then obtain

$$J_1 = c_1(2\rho)^n n^{n+\frac{1}{2}\alpha-\frac{1}{2}+\alpha\omega} e^{\beta\sqrt{2n}} \exp\bigg(-\frac{x^2}{2} + \frac{\sqrt{3}\big(\pi^2\sigma^2-6\big)}{12\sqrt{\alpha}\pi\sigma}\,x\bigg)\Big(1 + O\big(n^{-\frac{1}{2}}\big)\Big),$$

where β and ρ are the same as in the Proposition 27 and

$$c_1 := \frac{\sqrt{3}}{\pi^{3/2}\sigma}\left(\frac{1}{\Gamma(1+\omega)}\sqrt{\frac{24}{\alpha\pi}}\right)^{\alpha}\left(\frac{12}{\alpha\pi^2}\right)^{\alpha\omega}.$$

Summing over k in the range $\frac{1}{\alpha}k_- \leqslant k \leqslant \frac{1}{\alpha}k_+$ and using the integral

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2+tx}\,\mathrm{d}x = \sqrt{2\pi}\,e^{\frac{1}{2}t^2} \qquad (t \in \mathbb{R}),$$

we then deduce (8.3). □

We now sketch the proof of Theorem 26. By the change of variables $e(z) = e^{y^2}$, which is locally invertible, we obtain

$$z \mapsto \zeta(y) = \frac{y}{\sqrt{e_2}} - \frac{e_3}{2e_2^2}\,y^2 + \frac{2e_2^3 - 4e_2e_4 + 5e_3^2}{8e_2^{7/2}}\,y^3 + \cdots,$$

for small $|y|$. Then

$$[z^n]d(z)^{k+\omega_0}\prod_{1\leqslant j\leqslant k}\left(e(z)^{j+\omega}-1\right)^{\alpha} = \frac{1}{2\pi i}\oint_{|y|=r} y^{-n-1}h_{n,k}(y)e^{ky}\prod_{1\leqslant j\leqslant k}\left(e^{(j+\omega)y^2}-1\right)^{\alpha}\mathrm{d}y,$$

where

$$h_{n,k}(y) := \left(\frac{\zeta(y)}{y}\right)^{-n-1}\left(e^{-y}d(\zeta(y))\right)^{k}\zeta'(y)d(\zeta(y))^{\omega_0},$$

with the last term satisfying $d(\zeta(y))^{\omega_0} = 1 + O(|y|)$. Since y is of order $n^{-\frac{1}{2}}$, we use the local expansion

$$\begin{aligned} h_{n,k}(y) = e_2^{\frac{1}{2}n}\exp\Bigg(&\frac{e_3}{2e_2^{3/2}}\,ny - \frac{e_2^3 - 2e_2e_4 + e_3^2}{4e_2^3}\,ny^2 \\ &+ \frac{d_1-1}{\sqrt{e_2}}\,ky - \frac{d_2e_3 - 2d_2e_2 + d_1^2e_2 - e_3}{2e_2^2}\,ky^2 + O\left(n|y|^3\right)\Bigg). \end{aligned}$$

Then Proposition 27 can be applied as long as $\beta > 0$, namely, (8.1) holds. The remaining analysis being similar to that of Theorem 21, we omit the details. □

Note that the proof can be extended to the situation when $\lambda_j = 0$ for $1 \leqslant j < m$ and $\lambda_m > 0$, $m \geqslant 2$.

8.2. **Self-dual Λ-Fishburn matrices with $\lambda_1 > 0$.** We now consider general self-dual Λ-Fishburn matrices with $\lambda_1 > 0$.

Lemma 28. *The generating function for self-dual Λ-Fishburn matrices is given by (z marking the matrix size)*

$$\sum_{k\geqslant 0}\Lambda(z)^{k+1}\prod_{1\leqslant j\leqslant k}\left(\Lambda\left(z^2\right)^j - 1\right).$$

We omit the proof, which follows directly from that for the case when $\Lambda = \mathbb{Z}_{\geqslant 0}$ given in [30].

Corollary 29. *If $\lambda_1 > 0$, then the number of self-dual Λ-Fishburn matrices of size n satisfies*

$$[z^n]\sum_{k\geqslant 0}\Lambda(z)^{k+1}\prod_{1\leqslant j\leqslant k}\big(\Lambda(z^2)^j-1\big) = c_0 e^{\beta\sqrt{n}}\rho^{\frac{1}{2}n}n^{\frac{1}{2}(n+1)}\big(1+O\big(n^{-\frac{1}{2}}\big)\big),$$

where $\beta = \frac{\sqrt{6\lambda_1}}{\pi}\log 2$, *and* $(c,\rho) = \Big(\frac{3\sqrt{2}}{\pi^{3/2}}\,2^{\frac{\lambda_2}{\lambda_1}-\frac{\lambda_1}{2}}e^{-\frac{\lambda_1}{4}-\frac{\pi^2}{24}+\frac{\pi^2\lambda_2}{12\lambda_1^2}+\frac{3\lambda_1}{2\pi^2}(\log 2)^2}, \frac{6\lambda_1}{e\pi^2}\Big)$.

Proof. Condition (8.1) holds because $d_1 > 0$ and $e_3 = 0$. Apply Theorem 26 with $\omega_0 = \alpha = 1$, $\omega = 0$, $d_1 = e_2 = \lambda_1$, $d_2 = e_4 = \lambda_2$. □

The asymptotic approximation of the corollary implies that if λ_1 is fixed, then no matter how many copies other positive integers are used as entries, the resulting asymptotic count of self-dual matrices of large size differs only in the leading constant.

In particular, we obtain (1.10) by substituting $\lambda_1 = \lambda_2 = 1$ in Corollary 29, and (1.11) by $\lambda_1 = 1$ and $\lambda_2 = 0$, respectively, on the asymptotics of non-primitive and primitive self-dual Fishburn matrices, respectively. The situation of prime number entries when $\Lambda(z) = 1 + z + \sum_{p \text{ prime}} z^p$ leads to the same dominant asymptotics as in (1.10).

We now examine the three statistics (first row-size, diagonal sum, and the number of 1s) on random self-dual Λ-Fishburn matrices, beginning with the corresponding bivariate generating functions. For convenience, we include the empty matrix with size 0.

Proposition 30 (Statistics on self-dual Λ-Fishburn matrices)**.** *For self-dual Λ-Fishburn matrices, we have the following bivariate generating functions with z marking the matrix size and v the respective statistics.*

(i) The size of the first row

$$\Lambda(vz)\sum_{k\geqslant 0}\Lambda(z)^k\prod_{1\leqslant j\leqslant k}\big(\Lambda\big(vz^2\big)\Lambda\big(z^2\big)^{j-1}-1\big), \tag{8.4}$$

(ii) the size of the diagonal

$$\Lambda(vz)\sum_{k\geqslant 0}\Lambda(z)^k\prod_{0\leqslant j< k}\big(\Lambda\big(v^2z^2\big)\Lambda\big(z^2\big)^{j}-1\big), \tag{8.5}$$

and

(iii) the number of 1*s*

$$\sum_{k\geqslant 0}(\Lambda(z)+\lambda_1(v-1)z)^{k+1}\prod_{1\leqslant j\leqslant k}\big(\big(\Lambda(z^2)+\lambda_1(v^2-1)z^2\big)^j-1\big). \tag{8.6}$$

The proof is omitted since it is very similar to that for Λ-Fishburn matrices (see Proposition 4), using the same ideas in [30] for enumerating self-dual matrices.

Theorem 31. *Assume $\lambda_1 > 0$ and that all self-dual Λ-Fishburn matrices of size n are equally likely to be selected. Then in a random matrix, the size X_n of the first row and the half of the diagonal size $\frac{1}{2}Y_n$ both satisfy a central limit theorem with logarithmic mean and variance:*

$$\frac{X_n-\log n}{\sqrt{\log n}} \xrightarrow{d} \mathscr{N}(0,1), \quad \textit{and} \quad \frac{\frac{1}{2}Y_n-\log n}{\sqrt{\log n}} \xrightarrow{d} \mathscr{N}(0,1),$$

and for the number of 1*s* Z_n*, if* $\lambda_2 > 0$*, then* $n - Z_n$ *is the convolution of two Poisson variates:*

$$n - Z_n \xrightarrow{d} 2\text{Poisson}\big(\tfrac{\lambda_2}{\lambda_1}\log 2\big) * 4\text{Poisson}\big(\tfrac{\lambda_2\pi^2}{12\lambda_1}\big),$$

while if $\lambda_2 = 0$*, then* $\mathbb{P}(Z_n = n) \to 1$.

Proof. We rely on Theorem 26, following the same ideas used in the proof of Theorem 25.

(i) The first row size: by the approximation (7.3), we apply first Theorem 26 to (8.4) with $\omega_0 = v$, $\omega = v-1$, $d_1 = e_2 = \lambda_1$, $d_2 = e_4 = \lambda_2$, $e_3 = 0$, and $\alpha = 1$, giving rise to an asymptotic approximation to $a_n\mathbb{E}\big(v^{X_n}\big)$. Then normalizing the resulting expression by a_n (or by the same expression with $v = 1$), we obtain

$$\mathbb{E}\big(v^{X_n}\big) = \frac{1}{\Gamma(v)}\Big(\frac{6}{\pi^2}\Big)^{v-1} n^{v-1}\big(1 + O\big(n^{-\frac{1}{2}}\big)\big),$$

uniformly for bounded v. The asymptotic normality (or Poisson$(\log n)$) then follows from Quasi-Powers theorem.

(ii) The diagonal size. Similarly, by (8.5), (7.3), and then Theorem 26, we obtain

$$\mathbb{E}\big(v^{Y_n}\big) = \frac{1}{\Gamma(v^2)}\Big(\frac{6}{\pi^2}\Big)^{v^2-1} n^{v^2-1}\big(1 + O\big(n^{-\frac{1}{2}}\big)\big),$$

uniformly for bounded v.

(iii) The number of 1s. In this case, Theorem 26 does not apply to (8.6) because $e_2 = \lambda_1 v^2$ is a complex number in general and $e_2 > 0$ may not hold. However, the proof there does apply by considering $e(z/\sqrt{e_2})$, similar to Theorem 21. The result is the same as if we apply formally Theorem 21 with $\omega_0 = \alpha = 1$, $\omega = 0$, $d_1 = \lambda_1 v$, $d_2 = \lambda_2$, $e_2 = \lambda_1 v^2$, $e_3 = 0$, $e_4 = \lambda_2$,

$$\mathbb{E}\big(v^{n-Z_n}\big) = 2^{\frac{\lambda_2}{\lambda_1}(v^2-1)} e^{\frac{\lambda_2\pi^2}{12\lambda_1^2}(v^4-1)}\big(1 + O\big(n^{-\frac{1}{2}}\big)\big),$$

where the first term on the right-hand side is the probability generating function of two Poisson distributions if $\lambda_2 > 0$. The right-side becomes 1 when $\lambda_2 = 0$.

□

8.3. **Asymptotics of Λ-Fishburn matrices whose smallest nonzero entry is 2.** We consider Fishburn matrices whose smallest nonzero entry is 2. We assume that there is at least an odd number in Λ, namely,

$$\lambda_1 = \cdots = \lambda_{2m-1} = 0, \text{ and } \lambda_2, \lambda_{2m+1} > 0, \tag{8.7}$$

for $m \geqslant 1$. Otherwise, if Λ contains only even numbers, then, by dividing all entries by 2, the corresponding asymptotics and distributional properties can be dealt with by the same framework of Section 5. It turns out that $m = 1$ ($\lambda_1 = 0$ but $\lambda_3 > 0$) and $m \geqslant 2$ have different asymptotic behaviors, and in the latter case the dependence on the parity of n is more pronounced, one technical reason being that the condition (8.1) fails when $m \geqslant 2$, and the odd case needs special treatment.

Lemma 32. *Given a formal power series* $B(z) = \sum_{n\geqslant 0} b_n z^n$ *with* $b_n \simeq c_0\rho_0^n n^{n+t}$*,* $\rho_0 \neq 0$*, we have, for even n,*

$$[z^{\frac{1}{2}n}]e^{\beta n z}B(z) \simeq c\rho^{\frac{1}{2}n}n^{\frac{1}{2}n+t}, \quad \textit{with} \quad (c,\rho) = \big(c_0 2^{-t}e^{\frac{2\beta}{e\rho_0}}, \tfrac{1}{2}\rho_0\big).$$

Proof. Expand $e^{n\beta z}$ at $z = \frac{2}{e\rho_0 n}$, the asymptotic saddle-point of $z^{-\frac{1}{2}n}B(z)$, and estimate the error as in Section 4.2. $\square$

Theorem 33 (Λ-Fishburn matrices with 2 as the smallest entries)**.** *Assume that Λ is a multiset of nonnegative integers satisfying (8.7) with $\Lambda(0) = 1$. If $m = 1$, then the number of Λ-Fishburn matrices of size n satisfies*

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left(1-\Lambda(z)^{-j}\right) = ce^{\beta\sqrt{n}}\rho^{\frac{1}{2}n}n^{\frac{1}{2}n+1}\left(1+O\left(n^{-\frac{1}{2}}\right)\right),$$

where $\beta = \frac{\lambda_3\pi}{2\sqrt{3}\lambda_2^{3/2}}$, and $(c,\rho) = \left(\frac{3\sqrt{6}}{\pi^2}\, e^{\frac{\pi^2}{6}\left(\frac{\lambda_4}{\lambda_2^2}-\frac{1}{2}-\frac{7\lambda_3^2}{8\lambda_2^3}\right)}, \frac{3\lambda_2}{e\pi^2}\right)$; and if $m \geqslant 2$, then

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left(1-\Lambda(z)^{-j}\right) = \begin{cases} c' e^{\beta\sqrt{n}}\rho^{\frac{1}{2}n}n^{\frac{1}{2}n+1}\left(1+O\left(n^{-\frac{1}{2}}\right)\right), & \text{if } n \text{ is even;}\\ c_m e^{\beta\sqrt{n}}\rho^{\frac{1}{2}n}n^{\frac{1}{2}n-m+\frac{5}{2}}\left(1+O\left(n^{-\frac{1}{2}}\right)\right), & \text{if } n \text{ is odd,}\end{cases} \tag{8.8}$$

where ρ and β remain the same, $c' = \frac{6\sqrt{6}}{\pi^2}\, e^{\frac{\pi^2}{6}\left(\frac{\lambda_4}{\lambda_2^2}-\frac{1}{2}\right)}$, and $c_m = \frac{\sqrt{2}\pi^{2m-3}}{3^{m-2}}\cdot\frac{\lambda_{2m+1}}{\lambda_2^{m+1/2}}\, e^{\frac{\pi^2}{6}\left(\frac{\lambda_4}{\lambda_2^2}-\frac{1}{2}\right)}$.

Note that c and c' differ by a factor of 2.

Proof. In either case, we rely on the generating function on the right-hand side of (2.5). When $m = 1$, we easily check condition (8.1) since $d_1 = 0$ and $e_3 > 0$, and we can apply Theorem 26 with $\omega_0 = 1, \alpha = 2, \omega = d_1 = 0$, and $e_j = \lambda_j$ for $j = 2,3,4$.

Then we consider the case when $m = 2$. Following the proof of Theorem 26, we begin with the change of variables $\Lambda(z) = e^{y^2}$ (since $\lambda_1 = 0$), which in the current setting leads to the local expansion for the inverse function $\zeta(y)$ satisfying $\Lambda(\zeta(y)) = e^{y^2}$:

$$\zeta(y) = \frac{y}{\lambda_2^{1/2}} + \frac{\lambda_2^2-2\lambda_4}{4\lambda_2^{5/2}}\,y^3 - \frac{\lambda_5}{2\lambda_2^3}\,y^4 + \frac{5\lambda_2^4-36\lambda_2^2\lambda_4-48\lambda_2\lambda_6+84\lambda_4^2}{96\lambda_2^{9/2}}\,y^5+\cdots,$$

when $|y| \sim 0$. Then, we have

$$a_n = \frac{\lambda_2^{\frac{1}{2}n}}{2\pi i}\oint_{|y|=r} y^{-n-1}h_n(y)\sum_{0\leqslant k\leqslant\lfloor\frac{1}{4}n\rfloor} e^{(k+1)y^2}\prod_{1\leqslant j\leqslant k}\left(e^{jy^2}-1\right)^2 \mathrm{d}y,$$

where

$$h_n(y) := \sqrt{\lambda_2}\left(\frac{\zeta(y)}{y/\sqrt{\lambda_2}}\right)^{-n-1}\zeta'(y) = \exp\left(-\frac{\lambda_2^2-2\lambda_4}{4\lambda_2^2}\,ny^2+\frac{\lambda_5}{2\lambda_2^{5/2}}\,ny^3+O\left(|y|^2+n|y|^4\right)\right). \tag{8.9}$$

Let $\beta := \frac{1}{2}\left(\frac{\lambda_4}{\lambda_2^2}-\frac{1}{2}\right)$. Now if n is even, we have

$$\begin{aligned} a_n &= \lambda_2^{\frac{1}{2}n}[y^n]e^{\beta ny^2}\left(1+O\left(|y|^2+n|y|^3\right)\right)\sum_{0\leqslant k\leqslant\lfloor\frac{1}{4}n\rfloor} e^{ky^2}\prod_{1\leqslant j\leqslant k}\left(e^{jy^2}-1\right)^2\\ &= \lambda_2^{\frac{1}{2}n}\left(1+O\left(n^{-\frac{1}{2}}\right)\right)[y^{\frac{1}{2}n}]e^{\beta ny}\sum_{k\geqslant 0}e^{ky}\prod_{1\leqslant j\leqslant k}\left(e^{jy}-1\right)^2, \end{aligned}$$

which, together with Theorem 21 and Lemma 32, leads to the even case of (8.8).

On the other hand, when n is odd, then since y is of order $n^{-\frac{1}{2}}$, we can rewrite $h_n(y)$ in (8.9) as

$$h_n(y) = e^{\beta n y^2}\Big(1 + \frac{\lambda_5}{2\lambda_2^{5/2}}\, n y^3 + O\big(|y|^2 + n|y|^4\big)\Big);$$

from this we identify the lowest odd power of y, and obtain

$$a_n = \frac{\lambda_5}{2\lambda_2^{5/2}}\, n\lambda_2^{\frac{1}{2}n}\big(1 + O\big(n^{-\frac{1}{2}}\big)\big)[y^{n-3}]e^{\beta n y^2} \sum_{0\leqslant k\leqslant n} e^{ky^2} \prod_{1\leqslant j\leqslant k} \big(e^{jy^2} - 1\big)^2,$$

whose asymptotic approximation then follows from (8.8) in the even case since $n-3$ is even.

In general, when $m \geqslant 3$, by splitting $\Lambda(z)$ into odd and even parts, using for example Lagrange's inversion formula in the form

$$[y^k]\zeta(y) = \frac{1}{k}[t^{k-1}]\Big(\frac{t}{\log\Lambda(t)}\Big)^k \qquad (k = 1, 2, \ldots),$$

we deduce that the smallest odd power of y in the Taylor expansion of $h_n(y)e^{-\beta n y^2}$ is given by

$$\frac{\lambda_{2m+1}}{2\lambda_2^{m+1/2}}\,(n - 2m + 1)\,y^{2m-1}.$$

The expression of c_m then follows from that in the even case. □

In particular, the number of Fishburn matrices without using 1 as entries ($\Lambda = Z_{\geqslant 0} \setminus \{1\}$) satisfies

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\bigg(1 - \Big(\frac{1-z}{1-z+z^2}\Big)^j\bigg) = c e^{\beta\sqrt{n}}\rho^{\frac{1}{2}n} n^{\frac{1}{2}n+1}\big(1 + O\big(n^{-\frac{1}{2}}\big)\big),$$

where $\beta = \frac{\pi}{2\sqrt{3}}$, and $(c, \rho) = \big(\frac{3\sqrt{6}}{\pi^2}\, e^{-\frac{\pi^2}{16}}, \frac{3}{e\pi^2}\big)$, which marks a significant difference with that containing 1 as entries, as given in (1.1).

On the other hand, asymptotics of Λ-row-Fishburn matrices can be similarly treated, and exhibits a very similar behavior.

8.4. **Statistics on Λ-Fishburn matrices whose smallest nonzero entry is 2.**

Based on the generating functions of Proposition 4, we now consider the behavior of a general random Λ-Fishburn matrix with 2 being the smallest nonzero entry.

Theorem 34. *Assume that Λ satisfies (8.7). If all Λ-Fishburn matrices of size n are equally likely to be selected, then, in a random matrix under this distribution, the size X_n of the first row and the diagonal size Y_n are both asymptotically normally distributed in the following sense:*

$$\frac{X_n - \log n}{\sqrt{\log n}} \xrightarrow{d} \mathscr{N}(0,1), \qquad \textit{and} \qquad \frac{Y_n - 2\log n}{\sqrt{2\log n}} \xrightarrow{d} \mathscr{N}(0,1),$$

while the limiting distribution of the number Z_n of occurrences of 2 depends on m: if $m = 1$, then

$$\frac{\frac{1}{3}(n - 2Z_n) - \tau\sqrt{n}}{\sqrt{\tau\sqrt{n}}} \xrightarrow{d} \mathscr{N}(0,1), \qquad \Big(\tau := \frac{\lambda_3\pi}{2\sqrt{3}\lambda_2^{3/2}}\Big), \tag{8.10}$$

and if $m \geqslant 2$, then

$$Z_n^* \xrightarrow{d} \text{Poisson}\left(\tfrac{\lambda_4 \pi^2}{6\lambda_2}\right),$$

where

$$Z_n^* := \begin{cases} \frac{1}{2}\left(\frac{1}{2}n - Z_n\right), & \textit{if } n \textit{ is even;} \\ \frac{1}{2}\left(\frac{1}{2}(n-2m-1) - Z_n\right), & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. When $m = 1$, we rely on Theorem 26, following the same ideas used in the proof of Theorem 25, and when $m \geqslant 2$, the proof is similar to that of Theorem 33.

(i) Assume $m = 1$. For the first row sum, we have, by the generating function (2.11), the approximation (7.3) and a modification of the proof of Theorem 26,

$$[z^n]\Lambda(vz)\sum_{k\geqslant 0}\Lambda(z)^k \prod_{1\leqslant j\leqslant k}\left(\left(\Lambda(vz)\Lambda(z)^{j-1}-1\right)\left(\Lambda(z)^j-1\right)\right) = c(v)\rho^{\frac{1}{2}n} n^{\frac{1}{2}n+v}\left(1+O\left(n^{-\frac{1}{2}}\right)\right),$$

where $(c(v),\rho) = \left(\frac{\sqrt{6}}{\Gamma(v)}\left(\frac{3}{\pi^2}\right)^v e^{\frac{\pi^2}{6}\left(\frac{\lambda_4}{\lambda_2^2}-\frac{7\lambda_3^2}{8\lambda_2^3}-\frac{1}{2}\right)}, \frac{3\lambda_2}{e\pi^2}\right)$. Thus

$$\mathbb{E}\left(v^{X_n}\right) = \frac{1}{\Gamma(v)}\left(\frac{3}{\pi^2}\right)^{v-1} n^{v-1}\left(1+O\left(n^{-\frac{1}{2}}\right)\right), \tag{8.11}$$

uniformly for bounded v. Then the asymptotic normality (or Poisson($\log n$)) follows from Quasi-Powers theorem. When $m = 2$, we follow the same procedure as in the proof of Theorem 33 and obtain

$$\begin{aligned}[z^n]&\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left(1-\Lambda(vz)^{-1}\Lambda(z)^{1-j}\right) \\ &= \lambda_2^{\frac{1}{2}n}\left(1+\left(n^{-\frac{1}{2}}\right)\right)[y^{\frac{1}{2}n}]e^{\beta n y}\sum_{k\geqslant 0}e^{ky}\prod_{1\leqslant j\leqslant k}\left(e^{jy}-1\right)\left(e^{(j+v-1)y}-1\right),\end{aligned}$$

when n is even. By (7.9) and Lamma 32, we then deduce the same asymptotic approximation (8.11) when n is even. When n is odd, the corresponding asymptotic approximation differs by a factor of n^{-m} as in Theorem 33 but the resulting normalizing expression is still (8.11).

(ii) Very similarly, for the diagonal size, we apply first Theorem 26 to the generating function (2.12) when $m = 1$ with $\alpha = 2$, $\omega_0 = v$, $\omega = v-1$, $d_1 = 0$, $d_2 = e_2 = \lambda_2$, $e_3 = \lambda_3$ and $e_4 = \lambda_4$ and obtain an asymptotic approximation to the nth coefficient. Then normalizing the resulting expression gives

$$\mathbb{E}\left(v^{Y_n}\right) = \frac{1}{\Gamma(v)^2}\left(\frac{3}{\pi^2}\right)^{2(v-1)} n^{2(v-1)}\left(1+O\left(n^{-\frac{1}{2}}\right)\right),$$

uniformly for bounded v. The same expression remains true when $m \geqslant 2$ although the proof proceeds along the lines of that of Theorem 33.

(iii) The number of 2s is more involved. Consider first $m = 1$. Similar to (2.13) for the number of 1s, we now have the generating function

$$\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left(1-\left(\Lambda(z)+\lambda_2(v-1)z^2\right)^{-j}\right)$$
$$= \sum_{k\geqslant 0}\left(\Lambda(z)+\lambda_2(v-1)z^2\right)^{k+1}\prod_{1\leqslant j\leqslant k}\left(\left(\Lambda(z)+\lambda_2(v-1)z^2\right)^{j}-1\right)^2. \tag{8.12}$$

Then if $\lambda_3 > 0$, we get, by a similar modification of the proof of Theorem 26 (see Theorem 31), the Quasi-Powers approximations,

$$\mathbb{E}\left(v^{\frac{1}{2}n-Z_n}\right) = c(v)e^{\tau\sqrt{n}(v^{\frac{3}{2}}-1)}\left(1+O\left(n^{-\frac{1}{2}}\right)\right),$$

where τ is given in (8.10) and

$$c(v) := e^{-\frac{7\lambda_3^2\pi^2}{48\lambda_2^3}(v^3-1)+\frac{\lambda_4\pi^2}{6\lambda_2^2}(v^2-1)}.$$

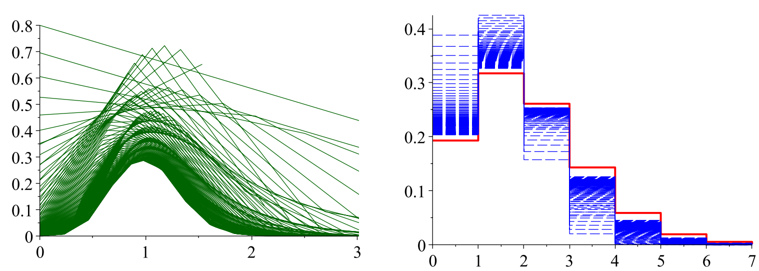
The asymptotic normality then results from the Quasi-Powers theorems; see [20, 27]. Indeed, Z_n^* is asymptotically Poisson distributed with parameter $\tau\sqrt{n}$.

When $m \geqslant 2$, we obtain, by (8.12), the change of variables $\Lambda(z)+(\lambda_2-1)vz^2 = e^{y^2}$, and modifying the proof of (8.8) (see also the proof of Theorem 31),

$$\mathbb{E}\left(v^{\frac{1}{2}n-Z_n}\right) = \begin{cases} e^{\frac{\lambda_4\pi^2}{6\lambda_2^2}(v^2-1)}, & \text{if } n \text{ is even;}\\ v^{m+\frac{1}{2}}e^{\frac{\lambda_4\pi^2}{6\lambda_2^2}(v^2-1)}, & \text{if } n \text{ is odd.}\end{cases}$$

This proves the Poisson limit law.

□



$\Lambda(z) = 1+z^2+z^3$ $\quad$ $\Lambda(z) = 1+z^2+z^4+z^5$

$\mathscr{N}\left(\frac{\pi\sqrt{n}}{2\sqrt{3}}, \frac{\pi\sqrt{n}}{2\sqrt{3}}\right)$ $\quad$ Poisson$\left(\frac{\pi^2}{6}\right)$

Figure 8.1. Histograms of the number of 2s in two different compositions of random Λ-Fishburn matrices. Left: the distributions $\mathbb{P}\left(\frac{1}{3}\left(\frac{n}{2}-Z_n\right) = \lfloor x\mu_n\rfloor\right)$ with μ_n denoting the exact mean, which is asymptotic to $\frac{\pi\sqrt{n}}{2\sqrt{3}}$; right: $\mathbb{P}\left(Z_n^* = k\right)$, where $Z_n^* := \frac{1}{2}\left(\frac{n}{2}-Z_n\right)$ when n is even, and $Z_n^* := \frac{1}{2}\left(\frac{n}{2}-2-Z_n\right)$ when n is odd, where the red line represents the corresponding Poisson distribution.

For random Λ-row-Fishburn matrices, one can derive very similar types of results: zero-truncated Poisson with parameter $\frac{\lambda_1}{\lambda_2}\log 2$ for the first row size, $\mathscr{N}(\log n, \log n)$ for the diagonal size, and $\mathscr{N}\left(\tau\sqrt{n}, \tau\sqrt{n}\right)$ with $\tau := \frac{\lambda_3\pi}{2\sqrt{6}\lambda_2^{3/2}}$ or Poisson($\frac{\lambda_4\pi^2}{12\lambda_2^2}$) limit law when $m = 1$ or $m \geqslant 2$, respectively, for the number of 2s.

9. Conclusions

Motivated by the asymptotic enumeration of and statistics on Fishburn matrices and their variants, we developed in this paper a saddle-point approach to computing the asymptotics of the coefficients of generating functions with a sum-of-product form, and applied it to several dozens of examples. The approach is not only useful for the usual large-n asymptotics but also effective in understanding the stochastic behaviors of random Fishburn matrices, with or without further constraints on the entries or on the structure of the matrices. In particular, we identified a simple yet general framework and showed its versatile usefulness in this paper. Many new asymptotic distributions of statistics on random matrices are derived in a systematic and unified manner, which in turn demand further structural interpretations; for example, since the normal approximations we derived in this paper can indeed all be approximated by Poisson distributions with parameters depending on n (equal to the asymptotic mean), a natural question is why Poisson laws with bounded or unbounded parameters is ubiquitous in the random Λ-Fishburn matrices.

Other frameworks will be examined in a follow-up paper. In addition to different sum-of-product patterns, we will also work out cases for which our approach in this paper does not directly apply. For example, we have not found transformations for the series (1.3) such that our saddle-point method works.

Finally, the rank (or dimension) of a random Λ-Fishburn matrices represents another important statistic on random matrices, which is expected to follow a central limit theorem with linear mean and variance. Indeed, if we replace e^z in Remark 2 by $1/(1-z)$, then the local limit theorem given there still holds, which can be interpreted as the distribution of the dimension in a random row-Fishburn matrix of size n. The situation for random Fishburn matrices is however less clear as we do not have a simple decomposition as in row-Fishburn ones. This and related quantities will be investigated and clarified elsewhere.

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References

[1] E. Andresen and K. Kjeldsen. On certain subgraphs of a complete transitively directed graph. *Discrete Math.*, 14(2):103–119, 1976.

[2] George E. Andrews. Ramanujan's "lost" notebook. V. Euler's partition identity. *Adv. in Math.*, 61(2):156–164, 1986.

[3] George E. Andrews, Freeman J. Dyson, and Dean Hickerson. Partitions and indefinite quadratic forms. *Invent. Math.*, 91(3):391–407, 1988.

[4] George E. Andrews and Vít Jelínek. On q-series identities related to interval orders. *European J. Combin.*, 39:178–187, 2014.

[5] George E. Andrews, Jorge Jiménez-Urroz, and Ken Ono. q-series identities and values of certain L-functions. *Duke Math. J.*, 108(3):395–419, 2001.
[6] Mireille Bousquet-Mélou, Anders Claesson, Mark Dukes, and Sergey Kitaev. $(2+2)$-free posets, ascent sequences and pattern avoiding permutations. *J. Combin. Theory Ser. A*, 117(7):884–909, 2010.
[7] Graham Brightwell and Mitchel T Keller. Asymptotic enumeration of labelled interval orders. *arXiv preprint arXiv:1111.6766*, 2011.
[8] Kathrin Bringmann, Yingkun Li, and Robert C. Rhoades. Asymptotics for the number of row-Fishburn matrices. *European J. Combin.*, 41:183–196, 2014.
[9] Anders Claesson and Svante Linusson. $n!$ matchings, $n!$ posets. *Proc. Amer. Math. Soc.*, 139(2):435–449, 2011.
[10] H. Cohen. q-identities for Maass waveforms. *Invent. Math.*, 91(3):409–422, 1988.
[11] Ovidiu Costin and Stavros Garoufalidis. Resurgence of the Kontsevich-Zagier series. *Ann. Inst. Fourier (Grenoble)*, 61(3):1225–1258, 2011.
[12] Mark Dukes, Vít Jelínek, and Martina Kubitzke. Composition matrices, $(\mathbf{2}+\mathbf{2})$-free posets and their specializations. *Electron. J. Combin.*, 18(1):Paper 44, 9, 2011.
[13] Mark Dukes, Sergey Kitaev, Jeffrey Remmel, and Einar Steingrímsson. Enumerating $(2+2)$-free posets by indistinguishable elements. *J. Comb.*, 2(1):139–163, 2011.
[14] Mark Dukes and Robert Parviainen. Ascent sequences and upper triangular matrices containing non-negative integers. *Electron. J. Combin.*, 17(1):Research Paper 53, 16, 2010.
[15] Nathan J. Fine. *Basic hypergeometric series and applications*, volume 27. American Mathematical Society, Providence, RI, 1988.
[16] Peter C. Fishburn. Intransitive indifference with unequal indifference intervals. *J. Mathematical Psychology*, 7:144–149, 1970.
[17] Peter C Fishburn. Interval lengths for interval orders: A minimization problem. *Discrete Math.*, 47:63–82, 1983.
[18] Peter C. Fishburn. *Interval Orders and Interval Graphs*. John Wiley & Sons, Ltd., Chichester, 1985.
[19] Philippe Flajolet, Xavier Gourdon, and Philippe Dumas. Mellin transforms and asymptotics: harmonic sums. *Theoret. Comput. Sci.*, 144(1-2):3–58, 1995. Special volume on mathematical analysis of algorithms.
[20] Philippe Flajolet and Robert Sedgewick. *Analytic combinatorics*. Cambridge University Press, Cambridge, 2009.
[21] Shishuo Fu, Emma Yu Jin, Zhicong Lin, Sherry HF Yan, and Robin DP Zhou. A new decomposition of ascent sequences and Euler-Stirling statistics. *J. Combin. Theory Ser. A*, 170:105–141, 2020.
[22] F. G. Garvan. Congruences and relations for r-fishburn numbers. *J. Combin. Theory Ser. A*, 134:147–165, 2015.
[23] T. L. Greenough. *Representation and enumeration of interval orders and semiorders*. PhD thesis, Dartmouth College, 1976.
[24] P. E. Haxell, J. J. McDonald, and S. K. Thomason. Counting interval orders. *Order*, 4(3):269–272, 1987.
[25] Kazuhiro Hikami. q-series and L-functions related to half-derivatives of the Andrews-Gordon identity. *Ramanujan J.*, 11(2):175–197, 2006.
[26] Hsien-Kuei Hwang. *Théorèmes limites pour les structures combinatoires et les fonctions arithmétiques*. PhD thesis, LIX, Ecole polytechnique, 1994.
[27] Hsien-Kuei Hwang. On convergence rates in the central limit theorems for combinatorial structures. *European J. Combin.*, 19(3):329–343, 1998.
[28] Hsien-Kuei Hwang. Asymptotics of poisson approximation to random discrete distributions: an analytic approach. *Advances in Applied Probability*, 31(2):448–491, 1999.
[29] OEIS Foundation Inc. The on-line encyclopedia of integer sequences.
[30] Vít Jelínek. Counting general and self-dual interval orders. *J. Combin. Theory Ser. A*, 119(3):599–614, 2012.
[31] Vít Jelínek. Catalan pairs and Fishburn triples. *Adv. in Appl. Math.*, 70:1–31, 2015.
[32] Soheir Mohamed Khamis. Exact counting of unlabeled rigid interval posets regarding or disregarding height. *Order*, 29(3):443–461, 2012.
[33] Sergey Kitaev and Jeffrey Remmel. Enumerating $(\mathbf{2}+\mathbf{2})$-free posets by the number of minimal elements and other statistics. *Discrete Appl. Math.*, 159(17):2098–2108, 2011.
[34] Sergey Kitaev and Jeffrey B. Remmel. A note on p-ascent sequences. *J. Comb.*, 8(3):487–506, 2017.
[35] Paul Levande. Fishburn diagrams, Fishburn numbers and their refined generating functions. *J. Combin. Theory Ser. A*, 120(1):194–217, 2013.

[36] F. W. J. Olver. *Asymptotics and special functions*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1974. Computer Science and Applied Mathematics.
[37] Ken Ono. *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, volume 102 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2004.
[38] Boris Pittel. Paths in a random digital tree: limiting distributions. *Adv. in Appl. Probab.*, 18(1):139–155, 1986.
[39] A. Stoimenow. Enumeration of chord diagrams and an upper bound for Vassiliev invariants. *J. Knot Theory Ramifications*, 7(1):93–114, 1998.
[40] G. Szekeres. Some asymptotic formulae in the theory of partitions. II. *Quart. J. Math., Oxford Ser. (2)*, 4:96–111, 1953.
[41] K. P. Bogart T. L. Greenough. The representation and enumeration of interval orders. 1976.
[42] R. Wong. *Asymptotic approximations of integrals*, volume 34 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001. Corrected reprint of the 1989 original.
[43] Sherry H. F. Yan. On a conjecture about enumerating $(2+2)$-free posets. *European J. Combin.*, 32(2):282–287, 2011.
[44] Don Zagier. Vassiliev invariants and a strange identity related to the Dedekind eta-function. *Topology*, 40(5):945–960, 2001.
[45] Don Zagier. Quantum modular forms. In *Quanta of maths*, volume 11 of *Clay Math. Proc.*, pages 659–675. Amer. Math. Soc., Providence, RI, 2010.

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