ASYMPTOTICS AND STATISTICS ON FISHBURN MATRICES: DIMENSION DISTRIBUTION AND A CONJECTURE OF STOIMENOW

HSIEN-KUEI HWANG, EMMA YU JIN AND MICHAEL J. SCHLOSSER

Abstract. We establish the asymptotic normality of the dimension of large-size random Fishburn matrices by a complex-analytic approach. The corresponding dual problem of size distribution under large dimension is also addressed and follows a quadratic type normal limit law. These results represent the first of their kind and solve two open questions raised in the combinatorial literature. They are presented in a general framework where the entries of the Fishburn matrices are not limited to $\{0, 1\}$ or nonnegative integers $\mathbb{N}_0$. The analytic saddle-point approach we apply, based on a powerful transformation for q-series due to Andrews and Jelínek, is also useful in solving a conjecture of Stoimenow in Vassiliev invariants.

Keywords: Fishburn matrices; dimension; saddle-point method; q-series; a conjecture of Stoimenow.

1. Introduction and main results

Fishburn matrices (abbreviated as FMs), introduced by Peter Fishburn in 1970 during his study of interval orders [13], are upper-triangular square matrices with nonnegative integers as entries such that no row and no column contains exclusively zeros. The idea of interval orders can be traced back to Wiener's 1914 paper [28]; see also [12]. FMs also appeared a few years later under a different guise in the study of transitively directed graphs by Andresen and Kjeldsen [1], where essentially a recursive formula was given on the number of *primitive FMs* (FMs with entries 0 or 1) with respect to the dimension and the first row sum (which is $\xi(n, k)$ in [1]; see also Section 5.2). For example, all FMs with *size* (or sum of all entries) equal to 4 are depicted in Figure 1.1 and all primitive FMs of dimension 3 in Figure 1.2.

Apart from the connection between primitive FMs and transitively directed graphs, it is now known that FMs are in bijection with *interval orders, (**2**+**2**)-free posets, ascent sequences, certain pattern-avoiding permutations* and *regular linearized chord diagrams (regular LCDs)*, etc.; see for instance [6, 9, 11, 14, 20].

The numbers of FMs of a given size are known as the *Fishburn numbers* (see [25, A022493]), which can be computed by the Taylor coefficients of the generating function

$$\sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \big(1-(1-z)^j\big) = 1 + z + 2z^2 + 5z^3 + 15z^4 + 53z^5 + 217z^6 + \cdots . \tag{1.1}$$

This formal generating function was derived by Zagier [30], using a recursive formula found earlier by Stoimenow [27] for the number of regular LCDs with a given length; we postpone the exact definition of LCDs and regular LCDs to Section 4. Stoimenow also made in the same paper [27]

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$$\begin{pmatrix}4\end{pmatrix} \begin{pmatrix}1&2\\&1\end{pmatrix} \begin{pmatrix}2&1\\&1\end{pmatrix} \begin{pmatrix}1&1\\&2\end{pmatrix} \begin{pmatrix}2&0\\&2\end{pmatrix} \begin{pmatrix}3&0\\&1\end{pmatrix} \begin{pmatrix}1&0\\&3\end{pmatrix}$$

$$\begin{pmatrix}1&1&0\\&1&0\\&&1\end{pmatrix} \begin{pmatrix}1&0&1\\&1&0\\&&1\end{pmatrix} \begin{pmatrix}1&0&0\\&1&1\\&&1\end{pmatrix} \begin{pmatrix}2&0&0\\&1&0\\&&1\end{pmatrix} \begin{pmatrix}1&0&0\\&2&0\\&&1\end{pmatrix} \begin{pmatrix}1&0&0\\&1&0\\&&2\end{pmatrix} \begin{pmatrix}1&1&0\\&0&1\\&&1\end{pmatrix} \begin{pmatrix}1&0&0&0\\&1&0&0\\&&1&0\\&&&1\end{pmatrix}$$

Figure 1.1. *All* 15 *FMs of size* $n = 4$*. The average dimension of these matrices is* $\frac{1}{15}(1 \cdot 1 + 2 \cdot 6 + 3 \cdot 7 + 4 \cdot 1) \approx 2.533$*, which is already close to our asymptotic approximation* $\frac{6}{\pi^2}n = \frac{24}{\pi^2} \approx 2.432$ *in Theorem 1.*

$$\begin{pmatrix}1&0&0\\&1&0\\&&1\end{pmatrix} \begin{pmatrix}1&1&0\\&1&0\\&&1\end{pmatrix} \begin{pmatrix}1&0&0\\&1&1\\&&1\end{pmatrix} \begin{pmatrix}1&0&1\\&1&0\\&&1\end{pmatrix} \begin{pmatrix}1&1&0\\&1&1\\&&1\end{pmatrix} \begin{pmatrix}1&1&1\\&1&0\\&&1\end{pmatrix} \begin{pmatrix}1&0&1\\&1&1\\&&1\end{pmatrix} \begin{pmatrix}1&1&1\\&1&1\\&&1\end{pmatrix} \begin{pmatrix}1&1&0\\&0&1\\&&1\end{pmatrix} \begin{pmatrix}1&1&1\\&0&1\\&&1\end{pmatrix}$$

Figure 1.2. *All* 10 *primitive FMs of dimension* $n = 3$*. The average size (sum of entries) of these matrices is* $\frac{1}{10}(3 \cdot 1 + 4 \cdot 4 + 5 \cdot 4 + 6 \cdot 1) = 4.5$ *while the asymptotic average size equals* $\frac{1}{4}n(n + 1) = 3$ *in Theorem 3.*

a conjecture concerning the asymptotic relation between the Fishburn numbers and the number of connected regular LCDs of size n, which will be addressed in more detail at the end of this section.

Since the seminal work [6] by Bousquet-Mélou, Claesson, Dukes and Kitaev, much attention has been drawn to the refined enumeration of Fishburn structures with respect to various classical statistics; see for instance [6, 11, 17, 20, 21, 22, 23, 24]. Two types of statistics among all members of the Fishburn family are Eulerian and Stirling statistics [17]: any statistic whose distribution over a member of the Fishburn family equals the distribution of the dimension (resp. the first row sum) on FMs is called an *Eulerian* (resp. a *Stirling*) statistic; see Table 1 for a summary of the equidistributed Eulerian and Stirling statistics on seven Fishburn structures.

While there is a large literature on the combinatorial aspects of statistics over Fishburn structures, very few studies have been conducted on asymptotic and stochastic properties concerning structures of large size; see [8, 30] and our previous paper [19]. Questions such as (see [20]) "*what is the expected dimension of a random FM of size n when each of the size-n FMs is chosen with the same probability?*" and "*what is the expected size of a random FM when all FMs of the same dimension are equally likely?*" have remained open, and the primary purpose of this paper is to answer these questions in a more *complete* (including the variance and the limiting distribution) and more *systematic* (covering a wide class of generalized FMs) way.

In contrast to the Stirling statistics worked out in [19], which have typically *logarithmic behaviors* (logarithmic mean and logarithmic variance), the Eulerian statistics studied in this paper, namely, dimension distribution with fixed size, have asymptotically *linear mean and linear variance* (the corresponding dual statistic, size distribution of fixed dimension, is *quadratic*). Such a contrast is well known for statistics on permutations, but has remained mostly elusive on Fishburn structures.

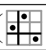
Fishburn structures	Eulerian statistics	Stirling statistics
FMs	dimension – 1	sum of the first row (or the last column) number of weakly northeast cells
($\mathbf{2}+\mathbf{2}$)-free posets	magnitude – 1	number of minimal elements
interval orders	magnitude – 1	number of minimal elements
Ascent sequences	asc, rep	zero, max, rmin
($\mathbf{2}-\mathbf{1}$)-avoiding sequences	rep	max
()-avoiding permutations	des, iasc	lmin, lmax, rmax
Regular linearized chord diagrams	length of the initial run of openers	number of pairs of arcs $(a,b),(c,d)$ such that $a<b=c-1<d-1$

Table 1. *Equidistributed Eulerian and Stirling statistics on Fishburn structures: statistics in the second (resp. third) and column are all equidistributed with each other; see* [6, 11, 17, 20, 24] *for the precise definitions.*

Whichever the case, the limiting distribution of these statistics are normal as long as the variances go unbounded, although the proof technicalities differ.

Since the entries of an FM of a given size can be viewed as an integer partition (but allowing 0 as entries) arranged on an upper-triangular matrix, there is yet a third class of Poisson statistics examined in detail in [19]: the number of occurrences of the smallest nonzero entry in the matrix. Similar to the classical integer partitions where 1 has a predominant frequency (see for instance [16]), the smallest nonzero entry in FMs appears almost everywhere. But different from the exponential limit law of the occurrences of the smallest part in random integer partitions, the smallest entry in FMs has its occurrences following mostly (but not always) a Poisson limit law; see [19]. This indicates an even higher concentration of the smallest entry near its expected value in the context of random FMs. Such a viewpoint will also be useful in interpreting our asymptotic results in this paper.

The approach developed in [19] relies on a direct two-stage saddle-point method that is applied to the generating functions with a sum-of-product form, and is very powerful in that it is not only applicable to the asymptotics of a wide class of concrete examples, but also provides an effective means of understanding the limit laws of Stirling statistics. In the present paper, we further extend the same saddle-point approach to Eulerian statistics. This extension is however not straightforward as a direct application fails due to the violent fluctuations in summing the dominant terms, similar to the summands on the left-hand side of (1.1). It turns out that a key property needed is a generalized Rogers-Fine identity derived by Andrews and Jelínek in [2]. Furthermore, an additional difficulty arises in handling the uniformity in the extra parameter of the probability generating function.

Given any multiset Λ of nonnegative integers with the generating function

$$\Lambda(z) = 1 + \lambda_1 z + \lambda_2 z^2 + \cdots, \tag{1.2}$$

a Λ-*FM* is an FM with entry multiset $\Lambda = \{0, \underbrace{1,\dots,1}_{\lambda_1 \text{ times}}, \underbrace{2,\dots,2}_{\lambda_2 \text{ times}}, \dots\}$. The original FMs correspond to the situation when all λ_j's equal 1, and the primitive FMs to $\lambda_j = \delta_{j,1}$, $j \geqslant 1$, the Kronecker symbol. Although such a matrix formulation requires that all the coefficients λ_j be nonnegative integers, our proof is independent of this restriction and the λ_j's can indeed be any nonnegative reals.

It is known that if $\Lambda(z)$ is analytic at $z = 0$ with $\lambda_1 > 0$ then the number of Λ-FMs of size n is given by (see [19, Corollary 20])

$$a_n := [z^n] \sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \big(1 - \Lambda(z)^{-j}\big) = cn^{\frac{1}{2}}(\lambda_1\mu)^n n!\big(1 + O\big(n^{-1}\big)\big), \tag{1.3}$$

where $(c, \mu) := \Big(\frac{12\sqrt{3}}{\pi^{5/2}} e^{\frac{\pi^2}{6}\left(\frac{\lambda_2}{\lambda_1^2} - \frac{1}{2}\right)}, \frac{6}{\pi^2}\Big)$. Here $[z^n]f(z)$ denotes the Taylor coefficient of $f(z)$. We see that the dominant asymptotic order (neglecting the leading constant c) depends crucially on λ_1, but not on any other λ_j's with $j \geqslant 2$, showing roughly the pervasiveness of 1 in a typical Λ-FM. On the other hand, the expression of c, as well as the violent cancellations of terms when summing the Taylor expansions of the finite products on the left-hand side of (1.3), implicitly points to the difficulty of the analysis involved; see [19] for more precise results.

1.1. **Dimension distribution of fixed-size FMs.** Define the bivariate generating function

$$F(z, v) := \sum_{n\geqslant 0} P_n(v)z^n = \sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \Big(1 - \frac{1}{1 + v(\Lambda(z)^j - 1)}\Big), \tag{1.4}$$

as an extension of the generating function in (1.3), where $P_n(v)$ is the generating polynomial of the dimension of size-n Λ-FMs with $P_n(1) = a_n$. The details to obtain (1.4) is provided in subsection 2.1.

Theorem 1 (Open problem of [8])**.** *Assume that $\Lambda(z)$ is analytic at $z = 0$ with $\lambda_1 > 0$ and that all Λ-FMs of size n are equally likely to be selected. Then the dimension X_n of a random matrix is asymptotically normally distributed with mean and variance both linear in n, namely,*

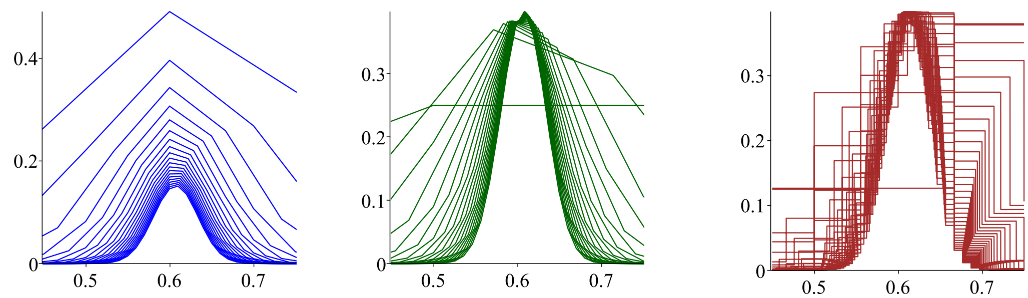
$$\frac{X_n - \mu n}{\sigma\sqrt{n}} \xrightarrow{d} \mathscr{N}(0, 1), \quad \textit{with} \quad (\mu, \sigma^2) := \Big(\frac{6}{\pi^2}, \frac{3(12 - \pi^2)}{\pi^4}\Big), \tag{1.5}$$

where the symbol $\xrightarrow{d}$ stands for convergence in distribution and $\mathscr{N}(0, 1)$ the standard normal distribution.

Remark 1. *The condition $\lambda_1 > 0$ was simply motivated by combinatorial applications: in case when $\lambda_1 < 0$, a change of variables gives an extra factor $(-1)^n$ for the nth Taylor coefficient and our analysis extends to such a case. The case $\lambda_1 = 0$ is discussed in Theorem 19.*

See Figure 1.3 for three different graphic renderings of the histograms of X_n when $\Lambda = \mathbb{N}$. Note that $\sigma^2 = \mu^2 - \frac{1}{2}\mu$, and the coefficient pair (μ, σ^2) is to some extent universal as we will also see its occurrences in other classes of FMs (albeit in slightly different scales).

What is particularly remarkable here is that the central limit theorem (1.5) is *independent* of Λ (as long as $\lambda_1 > 0$). The same also holds true for the first row sum (see [19]), which behaves asymptotically like a normal distribution with both mean and variance asymptotic to $\log n$. Such an "asymptotic invariance property" (1.5) may seem more surprising than its logarithmic



$\mathbb{P}(X_n = tn)$ $\qquad$ $\sqrt{\mathbb{V}(X_n)}\,\mathbb{P}(X_n = t\mathbb{E}(X_n))$ $\qquad$ $\sqrt{\mathbb{V}(X_n)}\,\mathbb{P}(X_n = \lfloor t\mathbb{E}(X_n)\rfloor)$

Figure 1.3. *Three different ways of visualizing the asymptotic normality of X_n where we plot the histograms of X_n in the case when $\Lambda(z) = (1-z)^{-1}$: for $n = 5j$, $1 \leqslant j \leqslant 20$ (left and middle) and $n = 3k$, $1 \leqslant k \leqslant 33$ (right).*

counterpart because linear statistics cover stochastically a wider range of variations. We can view this phenomenon from a few different angles.

First, from the asymptotic approximation (1.3), we see that the number of general Λ-FMs with $\lambda_1 > 0$ behaves roughly (modulo the leading constant c) like λ_1^n times the number of primitive FMs of the same size with $\Lambda(z) = 1 + z$. So we next examine more closely how the magic constant μ appears in random primitive FMs. The number of primitive FMs of size n is given by (see [25, A138265])

$$(a_n)_{n\geqslant 1} = (1, 1, 2, 5, 16, 61, 271, 1372, 7795, 49093, 339386, 2554596, \dots),$$

and it turns out that in this special case, we have an unexpected *identity* for the expected dimension:

$$\mu_n := \mathbb{E}(X_n) = \frac{a_{n+1}}{a_n} \qquad (n \geqslant 1); \tag{1.6}$$

see Section 3 for a more general form as well as a combinatorial proof of (1.6); in other words, *the sum of the dimensions of all size-n primitive FMs equals the number of size-$(n+1)$ primitive FMs.* In view of (1.3) and (1.6), we immediately get the asymptotic linearity of $\mathbb{E}(X_n)$ with the mean constant μ. In a similar manner, the second moment (and then the variance σ^2) can be approached via the same analytic and combinatorial arguments:

$$\sum_{1\leqslant k\leqslant n} \binom{k+1}{2} p_{n,k} + \sum_{1\leqslant k\leqslant n+1} k^2 p_{n+1,k} = a_{n+3},$$

where $p_{n,k}$ denotes the number of primitive FMs of size n and dimension k, which is $[v^k]P_n(v)$ from (1.4) when $\Lambda(z) = 1 + z$, appearing also in [25, A137252].

In addition, we will also derive finer asymptotic approximations for $\mathbb{E}(X_n)$ and $\mathbb{V}(X_n)$.

Theorem 2. *The mean and the variance of the dimension X_n (defined in Theorem 1) of a random Λ-FM of size n satisfy*

$$\mathbb{E}(X_n) = \mu\Big(n + \frac{3}{2}\Big) - \frac{\lambda_2}{\lambda_1^2} + O\big(n^{-1}\big), \tag{1.7}$$

$$\mathbb{V}(X_n) = \sigma^2\Big(n + \frac{3}{2}\Big) - \frac{1}{4} + \frac{\lambda_2}{2\lambda_1^2} + O\big(n^{-1}\big), \tag{1.8}$$

where (μ, σ^2) *is given in* (1.5).

Note that the dependence of $\mathbb{E}(X_n)$ and $\mathbb{V}(X_n)$ on Λ is weak: only the ratio of λ_2 and λ_1^2 appears in the constant terms, and similarly for higher central moments. For example,

$$\mathbb{E}\big(X_n - \mu_n\big)^3 = \frac{\pi^4 - 54\pi^2 + 432}{\pi^6}\Big(n + \frac{3}{2}\Big) + \frac{1}{12} - \frac{\lambda_2}{6\lambda_1^2} + O\big(n^{-1}\big),$$

$$\mathbb{E}\big(X_n - \mu_n\big)^4 = 3\mathbb{V}(X_n)^2 + \Big(6\sigma^4 - \frac{\mu^2}{12}\Big)n + O(1).$$

In principle, such calculations can be carried out further for all higher central moments and lead possibly to an alternative proof of (1.5) by the method of moments. But the cancellations involved in such a process are very heavy and complex, so we will instead work out an analytic, cancellation-free approach. Other λ_j's will appear in lower-order terms. We will also indicate in Section 3.3 the source of the seemingly strange but omnipresent ratio "$\frac{\lambda_2}{\lambda_1^2}$" in the second-order terms.

The same types of normal limit results are expected to hold for other classes of FMs, and we will briefly examine two of them: self-dual Λ-FMs (or persymmetric, namely, symmetric with respect to the anti-diagonal), and Λ-FMs whose smallest nonzero entries are 2. The corresponding central limit theorems are summarized in Table 2; see Section 6 for more information.

Λ-FMs with $\lambda_1 > 0$ (Theorem 1)	Self-dual Λ-FMs with $\lambda_1 > 0$ (Theorem 18)	Λ-FMs with $\lambda_1 = 0, \lambda_2 > 0$ (Theorem 19)
$\mathscr{N}(\mu n, \sigma^2 n)$	$\mathscr{N}(\mu n, 2\sigma^2 n)$	$\mathscr{N}\big(\frac{1}{2}\mu n, \frac{1}{2}\sigma^2 n\big)$

Table 2. *A summary of the central limit theorems for the dimension of different types of random Λ-FMs. Note specially the change in the mean and the variance coefficients: while the halving in the last column is well expected, the asymptotic doubling of the variance in the self-dual FMs comes as a little surprise.*

Finally, finer approximation results such as the optimal convergence rates (of order $n^{-\frac{1}{2}}$) or the corresponding local limit theorems are generally anticipated, but this direction lies outside the scope of the current paper.

1.2. **Size distribution of fixed-dimension FMs.** We now address a dual problem: the size distribution of random Λ-FMs with the same dimension. The problem is well-defined when Λ is finite and all coefficients of $\Lambda(z)$ are positive integers.

Theorem 3 (Extended open problem 5.5 of [20])**.** *Assume that $\Lambda(z)$ is a polynomial with positive coefficients and $\Lambda(1) \neq 1$, and that all Λ-FMs of dimension m are equally likely. Then the size Y_m of a random matrix is asymptotically normally distributed with mean and variance both of order $\Theta(m^2)$:*

$$\frac{Y_m - \hat{\mu}m^2}{\hat{\sigma}m} \xrightarrow{d} \mathscr{N}(0,1), \tag{1.9}$$

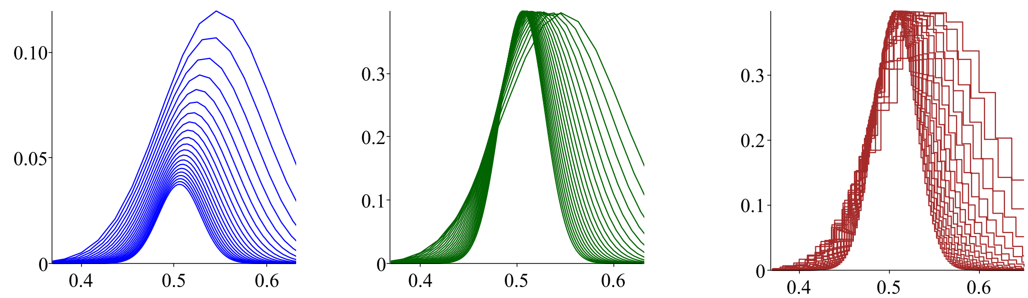
where $\hat{\mu}, \hat{\sigma}^2 > 0$ are given by

$$(\hat{\mu}, \hat{\sigma}^2) := \Big(\frac{\Lambda'(1)}{2\Lambda(1)}, \frac{1}{2}\Big(\frac{\Lambda'(1) + \Lambda''(1)}{\Lambda(1)} - \Big(\frac{\Lambda'(1)}{\Lambda(1)}\Big)^2\Big)\Big). \tag{1.10}$$

See Figure 1.4 for three different plots of the histograms of Y_m in the case of primitive FMs for which $(\hat{\mu}, \hat{\sigma}^2) = (\frac{1}{4}, \frac{1}{8})$. Note that, if $\Lambda(z) = 1 + \sum_{1 \leqslant j \leqslant \ell} \lambda_j z^j$ with $\ell \geqslant 1$ is a positive polynomial, then

$$2\hat{\sigma}^2 = \frac{1}{\Lambda(1)} \sum_{1 \leqslant j \leqslant \ell} \Big(j - \frac{\Lambda'(1)}{\Lambda(1)}\Big)^2 \lambda_j + \frac{(\Lambda'(1))^2}{\Lambda^3(1)} > 0.$$

Unlike the invariance property (1.5) and the limit laws in our previous paper [19], here the pair of constants $(\hat{\mu}, \hat{\sigma}^2)$ depends crucially on Λ.



$\mathbb{P}(X_n = tn)$ $\quad$ $\sqrt{\mathbb{V}(X_n)}\,\mathbb{P}(X_n = t\mathbb{E}(X_n))$ $\quad$ $\sqrt{\mathbb{V}(X_n)}\,\mathbb{P}(X_n = \lfloor t\mathbb{E}(X_n)\rfloor)$

Figure 1.4. *Three different ways of visualizing the asymptotic normality of Y_m where the histograms of Y_m are given in the case when $\Lambda(z) = 1 + z$: for $10 \leqslant n \leqslant 30$.*

Remark 2. *Define the random variable Y by $\mathbb{E}(z^Y) = \frac{\Lambda(z)}{\Lambda(1)}$. The quadratic behavior of Y_m naturally suggests the question: "*what is the probability that a randomly generated upper triangular matrix of dimension m is Fishburn when each entry is independently and identically distributed as Y (except for the upper-left and lower-right corners)?*" Our result implies particularly (see* (5.3)*) that in the primitive case (when Y is Bernoulli with mean $\frac{1}{2}$), the probability is asymptotic to*

$$4 \sum_{k \geqslant 0} (-1)^k 2^{-\binom{k+1}{2}} \sum_{0 \leqslant j \leqslant k} \prod_{1 \leqslant \ell \leqslant j} \frac{1}{1 - 2^{-\ell}} \approx 0.33359\ldots.$$

In other words, if we fix the two corners on the diagonal of the matrix to be 1, and generate all other entries by throwing an unbiased coin, consistently putting 0 or 1 as the entry according to the coin being head or tail, each independently of the others, then *more than one third of such matrices are Fishburn*.

1.3. **Asymptotic density of connected regular LCDs.** The proof of Theorem 1 is based on the saddle-point approach developed by the first two authors in [19] and a generalization of the Rogers-Fine identity due to Andrews and Jelínek [2], while Theorem 3 follows from a partial fraction decomposition and is simpler in nature.

It turns out that our saddle-point method is also useful in solving a conjecture of Stoimenow [27] that was subsequently reformulated by Zagier [30], where the enumeration of chord diagrams was studied in order to derive an upper bound for the dimension of the Vassiliev invariants space for knots. Based on numerical evidence, an asymptotic relation for the proportion of connected regular LCDs (among all regular LCDs) was then conjectured; see also [8, 30].

Theorem 4 (A conjecture in [27])**.** *Let f_n be the number of regular LCDs of size n (which equals the n-th Fishburn number), and g_n be that of connected regular LCDs of size n. Then*

$$\frac{g_n}{f_n} = e^{-1}\big(1 + O\big(n^{-1}\big)\big). \tag{1.11}$$

The same limit ratio is also observed for the derangement probability and the proportion of connected (ordinary) chord diagrams; see for example [5, 26] and [25, A068985].

Let $g(z) := \sum_{n\geqslant 1} g_n z^n$. Then the first few terms of $g(z)$ are given by (see [25, A022494])

$$g(z) = z + z^2 + 2z^3 + 5z^4 + 16z^5 + 63z^6 + 293z^7 + 1561z^8 + 9321z^9 + \cdots . \tag{1.12}$$

Our proof of (1.11) relies crucially on the functional equation obtained by Zagier in [30]:

$$\Phi(z, g(z)) = 1, \quad \text{with} \quad \Phi(z, v) := \frac{1}{1+v}\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\frac{1-(1-z)^j}{1+v(1-z)^j}, \tag{1.13}$$

together with a generalized Rogers-Fine identity derived by Andrews and Jelínek [2]. The function Φ is connected to F in (1.4) when $\Lambda(z) = (1-z)^{-1}$ by

$$F(z, v) = \frac{1}{v}\Phi\Big(z, \frac{1}{v} - 1\Big). \tag{1.14}$$

It is through this connection that our analytic techniques can be applied to solve the conjecture (1.11).

This paper is organized as follows. We prove in the next section the normal limit law of the dimension (Theorem 1). Then Theorem 2 concerning a more precise asymptotics of the expected dimension and the variance is shown in Section 3. Stoimenow's conjecture, which is now our Theorem 4, and the dual version of Theorem 1 are established in Section 4 and 5. Finally, we describe very briefly in Section 6 the limit results for the dimension in the self-dual case, and the case when $\lambda_1 = 0$, $\lambda_2 > 0$ and there exists at least one odd number in the entry-set.

Throughout this paper, the generic symbols $c, \varepsilon > 0$ always denote a constant and small quantity, respectively, whose values may not be the same at each occurrence. In contrast, the pair (μ, σ^2) always stands for the same value given in (1.5); we also define $q := \mu \log 2$. Furthermore, the notation $a_n \asymp b_n$ means that the ratio a_n/b_n remains bounded and not equal to zero as n tends to infinity. Finally, we set $\partial_z^m := \frac{\partial^m}{\partial z^m}$ for any $m \in \mathbb{Z}^+$.

2. Dimension distribution of random FMs

This section is devoted to proving Theorem 1, the central limit theorem for the dimension of random Λ-FMs of large size.

2.1. **A series transformation for the generating function (1.4).** We begin with seeking a series representation of $F(z, v)$ better than (1.4) because (1.4) contains negative coefficients in the Taylor expansion of $1 - \Lambda(z)^{-j}$. It is known from [17, 20] that the generating function of FMs with respect to size (variable z) and dimension (variable v) has two equivalent forms:

$$\sum_{k\geqslant 0} v^k \prod_{1\leqslant j\leqslant k}\frac{1-(1-z)^j}{v+(1-v)(1-z)^j} = 1 + \sum_{k\geqslant 1}\frac{v(1-z)^k}{1-v+v(1-z)^k}\prod_{1\leqslant j\leqslant k}\big(1-(1-z)^j\big).$$

Replacing $(1-z)^{-1}$ by $1+z$ on both sides gives two equivalent forms for the generating function of primitive FMs:

(2.1)
$$\sum_{k\geqslant 0} v^k \prod_{1\leqslant j\leqslant k} \frac{1-(1+z)^{-j}}{v+(1-v)(1+z)^{-j}} = 1 + \sum_{k\geqslant 1} \frac{v(1+z)^{-k}}{1-v+v(1+z)^{-k}} \prod_{1\leqslant j\leqslant k} \big(1-(1+z)^{-j}\big).$$

Since any FM with entry set Λ can be obtained by substituting each nonzero entry 1 in a primitive FM by an element from $\Lambda - \{0\}$, we substitute z by $\Lambda(z)-1$ on both sides of (2.1), implying that in addition to (1.4), the generating function of FMs with entry set Λ is also equal to

$$F(z,v) = 1 + \sum_{k\geqslant 1} \frac{v\Lambda(z)^{-k}}{1-v\big(1-\Lambda(z)^{-k}\big)} \prod_{1\leqslant j\leqslant k} \big(1-\Lambda(z)^{-j}\big), \tag{2.2}$$

but again the same negative sign problem occurs. A better expression for our purposes is the following one.

Lemma 5. *The bivariate generating function for the dimension of Λ-FMs satisfies*

$$F(z,v) = \sum_{n\geqslant 0} P_n(v) z^n = 1 - v + \sum_{k\geqslant 1} v^k \Lambda(z)^k \prod_{1\leqslant j<k} \frac{\big(\Lambda(z)^j - 1\big)^2}{1-(v-1)(\Lambda(z)^j-1)}, \tag{2.3}$$

where $P_n(v)$ is the generating polynomial of the dimension of Λ-FMs of size n.

Proof. We begin with the following identity of Andrews and Jelínek in [2, P. 186, first display]:

$$\sum_{k\geqslant 0} \frac{\big(\frac{s}{tx};x\big)_k \big(\frac{1}{y};\frac{1}{x}\big)_k}{(s;x)_k} t^k = y \sum_{k\geqslant 0} \frac{(y;x)_k (tx;x)_k}{(s;x)_k} x^k, \tag{2.4}$$

where $(a;z)_k := (1-a)(1-az)\cdots(1-az^{k-1})$. Substituting $t=1$, $s=\frac{(v-1)\Lambda(z)}{v}$, $x=y=\Lambda(z)$ gives

$$\sum_{k\geqslant 0} \frac{\big(\frac{v-1}{v};\Lambda(z)\big)_k \big(\frac{1}{\Lambda(z)};\frac{1}{\Lambda(z)}\big)_k}{(\frac{v-1}{v}\Lambda(z);\Lambda(z))_k} = \Lambda(z) \sum_{k\geqslant 0} \frac{(\Lambda(z);\Lambda(z))_k (\Lambda(z);\Lambda(z))_k}{(\frac{v-1}{v}\Lambda(z);\Lambda(z))_k} \Lambda(z)^k. \tag{2.5}$$

Simplifying these sums, we are then led to the following relations for the left-hand side (LHS) and the right-hand side (RHS):

$$\mathrm{LHS}(2.5) - 1 = v^{-1}(\mathrm{RHS}(2.2)-1), \quad \text{and} \quad \mathrm{RHS}(2.5) - 1 = v^{-1}(\mathrm{RHS}(2.3)-1).$$

This completes the proof. □

Interestingly, this lemma provides a combinatorial interpretation of a special case of the generalized Rogers-Fine identity (2.4), partially answering a question raised by Andrews and Jelínek [2].

2.2. **The exponential prototype.** We will follow mostly the approach developed in [19] but proceed differently, with particular efforts to handle the uniformity for the extra parameter v of the generating function. Following the same idea introduced in [19], we begin with the special case when $\Lambda(z) = e^z$; the general case will then be deduced by a change-of-variables argument.

Let

$$E(z, v) = 1 - v + \sum_{k \geqslant 0} E_k(z) R_k(z, v), \tag{2.6}$$

where

$$E_k(z) := e^{(k+1)z} \prod_{1 \leqslant j \leqslant k} \left(e^{jz} - 1\right)^2, \tag{2.7}$$

$$R_k(z, v) := v^{k+1} \prod_{1 \leqslant j \leqslant k} \frac{1}{1 - (v-1)\left(e^{jz} - 1\right)} = v \prod_{1 \leqslant j \leqslant k} \frac{1}{1 - (1 - v^{-1}) e^{jz}}..19 \tag{2.8}$$

The first few terms in the Taylor expansion of $E(z, v)$ are given by

$$E(z, v) = 1 + vz + (v + 2v^2)\frac{z^2}{2!} + (v + 12v^2 + 6v^3)\frac{z^3}{3!} + (v + 50v^2 + 132v^3 + 24v^4)\frac{z^4}{4!} + \cdots .$$

Here the coefficient of $\frac{v^k z^n}{n!}$ counts *the number of labelled* ***(2+2)*** *free posets of n elements with magnitude $k-1$*; see [7] and [25, A079144]. On the other hand, while all coefficients $[z^n v^k]E(z, v)$ are nonnegative, the individual terms $[z^n v^k]R_k(z, v)$ may still be negative; indeed, since

$$\frac{1}{1 - (v-1)(e^z - 1)} = \frac{e^{-z}}{1 + v(e^{-z} - 1)},$$

we see that $[z^n v^m]R_k(-z, -v)$ are nonnegative and $[z^n v^m]R_k(z, v) = (-1)^{n+m}[z^n v^m]R_k(-z, -v)$ may be negative.

Here we aim to prove the asymptotic normality of the random variable X_n, which, in the case of $\Lambda(z) = e^z$, is defined as

$$\mathbb{P}(X_n = k) := \frac{n![z^n v^k]E(z, v)}{n![z^n]E(z, 1)} = \frac{[z^n v^k]E(z, v)}{[z^n]E(z, 1)} \qquad (1 \leqslant k \leqslant n), \tag{2.9}$$

for $n \geqslant 1$, where X_n assumes only integer values. For that purpose, we will restrict our analysis to the range $|v - 1| = O(n^{-\frac{1}{2}})$, $v \in \mathbb{C}$, as $n \to \infty$.

Proposition 6. *For large n, the coefficient of z^n in the Taylor expansion of $E(z, v)$ defined in (2.6) satisfies*

$$[z^n]E\left(z, e^{i\theta}\right) = c n^{\frac{1}{2}} \mu^n n! e^{\mu n i\theta - \frac{1}{2}\sigma^2 n \theta^2}\left(1 + O\left((|\theta| + n|\theta|^3)\right)\right), \tag{2.10}$$

uniformly for $\theta = O(n^{-\frac{1}{2}})$, where $c = \frac{12\sqrt{3}}{\pi^{5/2}}$ and the pair (μ, σ^2) is defined in (1.5).

The approximation (2.10) implies, by (2.9), that

$$\mathbb{E}\left(e^{\frac{X_n - \mu n}{\sigma\sqrt{n}} i\vartheta}\right) = e^{-\frac{\mu}{\sigma}\sqrt{n}\, i\vartheta} \frac{[z^n]E\left(z, e^{\frac{i\vartheta}{\sigma\sqrt{n}}}\right)}{[z^n]E(z, 1)} \to e^{-\frac{1}{2}\vartheta^2}, \tag{2.11}$$

uniformly for $\vartheta = O(1)$. Then the asymptotic normality (1.5) of X_n (when $\Lambda(z) = e^z$) follows from the continuity theorem for characteristic functions; see [4, Theorem 26.3].

While the Taylor expansion of $R_k(z,v)$ (in z and v) still contains, in general, negative coefficients, the series (2.8) is suitable for our purposes because R_k plays asymptotically only a perturbative role when v is close to 1. This is roughly seen as follows. By the rightmost product expression in (2.8), we see, when $1 \leqslant k = O(n)$, $z \asymp n^{-1}$ and $|\theta| = o(1)$, that

$$R_k(z, e^{i\theta}) = e^{i\theta} \prod_{1\leqslant j\leqslant k} \frac{1}{1-(1-e^{-i\theta})e^{jz}} = O\Big(\prod_{1\leqslant j\leqslant k} (1+|\theta|)\Big) = e^{O(k|\theta|)}, \tag{2.12}$$

while, under the same setting, $|E_k(z)|$ will be of order $e^{k\log(k|z|)} = e^{O(n)}$ when $k \asymp n$; see (2.19).

2.3. **Ideas and outline of the proof of Proposition 6.** The key idea in the proof of (2.10) is first to show that not only the standard saddle-point approximation applies to the asymptotics of the individual terms $[z^n]E_k(z)R_k(z,v)$ when v is close to unity, but also the sum over k of all these terms is amenable by another application of the saddle-point method.

When realizing this idea using Cauchy's integral representation, a simple guiding heuristic to find out the dominant zones in $k \in [0, \lfloor \frac{1}{2}n \rfloor]$ and in $|z| = O(1)$, $|\arg z| \leqslant \pi$ is as follows. First, since $\theta = O(n^{-\frac{1}{2}})$, we focus (in view of (2.12)) on examining the contribution of

$$\sum_{0\leqslant k\leqslant \lfloor \frac{1}{2}n\rfloor} [z^n]E_k(z) \leqslant \sum_{0\leqslant k\leqslant \lfloor \frac{1}{2}n\rfloor} r^{-n}E_k(r), \tag{2.13}$$

for $r > 0$, where we are led to seek the pair (k,r) such that $r^{-n}E_k(r)$ is minimized in r (because it is an upper bound) but maximized in k (we need to pinpoint the dominant terms). Now when $r \asymp n^{-1}$ and $k \asymp n$, by the Euler-Maclaurin formula (see [19, Proposition 7]),

$$\begin{aligned} \log\big(r^{-n}E_k(r)\big) &= -n\log r + (k+1)r + 2\sum_{1\leqslant j\leqslant k} \log\big(e^{jr}-1\big) \\ &\sim J(k,r) := -n\log r + 2\int_1^k \log\big(e^{xr}-1\big)\,\mathrm{d}x, \end{aligned} \tag{2.14}$$

and we are led to solve the system of equations

$$\begin{cases} \dfrac{\partial}{\partial k} J(k,r) = 2\log\big(e^{kr}-1\big) = 0, \\ \dfrac{\partial}{\partial r} J(k,r) \sim -\dfrac{n}{r} + 2\displaystyle\int_0^k \frac{x}{1-e^{-xr}}\,\mathrm{d}x = 0. \end{cases}$$

The first equation implies that $kr = \log 2$; substituting this relation into the second equation gives (after a change of variables)

$$n \sim \frac{2k}{\log 2}\int_0^{\log 2} \frac{t}{1-e^{-t}}\,\mathrm{d}t = \frac{\pi^2}{6\log 2}k,$$

or, $k \sim qn$ with $q = \mu\log 2$, and, accordingly, $r \sim \frac{1}{\mu n}$. Then it is straightforward to check that $(\partial_k^2 J(k,r))(\partial_r^2 J(k,r)) - (\partial_k\partial_r J(k,r))^2 \sim -\frac{24}{\pi^2}n^2 < 0$. Thus $(k,r) = (qn, \frac{1}{\mu n})$ is a saddle-point we desired.

We will carry out the bivariate saddle-point heuristic sketched above to identify the dominant terms of $[z^n]E_k(z)R_k(z,v)$ via a two-stage analysis where k is first fixed and $r = |z|$ chosen to be the saddle-point in the corresponding Cauchy integral formula. Then we find out the range in

k where the terms attain their maximum values. For this purpose, we decompose the asymptotic evaluation of $[z^n]E(z,v)$ into three parts, with the dominant contribution coming from the last one:

$$[z^n]E(z,e^{i\theta}) = \sum_{0\leqslant k\leqslant\lfloor\frac12 n\rfloor}[z^n]E_k(z)R_k(z,e^{i\theta}) = T_1+T_2+T_3, \qquad (n\geqslant 1). \tag{2.15}$$

Here

$$\begin{aligned} T_1 &:= \Bigg(\sum_{0\leqslant k\leqslant k_-} + \sum_{k_+\leqslant k\leqslant\lfloor\frac12 n\rfloor}\Bigg)[z^n]E_k(z)R_k(z,e^{i\theta}),\\ T_2 &:= \Bigg(\sum_{k_-<k<k^-} + \sum_{k^+<k<k_+}\Bigg)[z^n]E_k(z)R_k(z,e^{i\theta}),\\ T_3 &:= \sum_{k^-\leqslant k\leqslant k^+}[z^n]E_k(z)R_k(z,e^{i\theta}), \end{aligned} \tag{2.16}$$

where $k_\pm := (\mu\log 2)n \pm \sqrt{2}\,\varsigma\, n^{\frac56}$ and $k^\pm := (\mu\log 2)n \pm \sqrt{2}\,\varsigma\, n^{\frac58}$ with

$$\varsigma^2 := \frac{3}{2\pi^4}\big(24(\log 2)^2-\pi^2\big). \tag{2.17}$$

While the choices of the coefficients q and ς^2 are crucial in identifying the largest terms in the summation of (2.15), the intervals $[0,k_-]$, (k_-,k^-), and (k^+,k_+), $[k_+,\lfloor\frac12 n\rfloor]$ here are simply taken for easier manipulation of the underlying saddle-point analysis. Note that $q=\mu\log 2\approx 0.42138$.

2.4. **Asymptotic approximations for** $\log E_k$ **and** $\log R_k$**.** We first derive an asymptotic approximation for $\log E_k(z)R_k(z,e^{i\theta})$ when $\theta = O(n^{-\frac12})$. Introduce the integral

$$I(z) := \int_0^z \frac{t}{1-e^{-t}}\,\mathrm{d}t = \frac{z^2}{2} + \operatorname{dilog}(e^{-z}), \tag{2.18}$$

where $\operatorname{dilog}(1-z) = -\int_0^z t^{-1}\log(1-t)\,\mathrm{d}t = \sum_{k\geqslant1}k^{-2}z^k$ denotes the dilogarithm function.

Lemma 7. *Assume $z\in\mathbb{C}$ and $z\neq 0$. We have*

$$\log E_k(z) = 2k\log(e^{kz}-1) - \frac{2I(kz)}{z} + \log\frac{2\pi(e^{kz}-1)}{z} + kz + O\big(k^{-1}+|z|\big), \tag{2.19}$$

$$\log R_k(z,e^{i\theta}) = \frac{(e^{kz}-1)i\theta}{z} - \frac{(e^{kz}-1)^2\theta^2}{4z} + O\big(|\theta| + |z|^{-1}|\theta|^3\big), \tag{2.20}$$

uniformly as $k\to\infty$, $|z|=o(1)$, $k|z|\leqslant 2\pi-\varepsilon$ and $\theta = o(|z|^{\frac13})$.

Proof. A direct application of Euler-Maclaurin summation formula (see [19, Proposition 7]) gives (2.19). On the other hand, the expansion

$$R_k\big(z,e^{i\theta}\big) = e^{i\theta}\prod_{1\leqslant j\leqslant k}\frac{1}{1-(1-e^{-i\theta})e^{jz}} = \exp\Bigg(i\theta + \sum_{l\geqslant1}\frac{(1-e^{-i\theta})^l}{l}\cdot\frac{e^{klz}-1}{1-e^{-lz}}\Bigg) \tag{2.21}$$

implies, by the estimate $1-e^{-lz}\sim lz$ for small $|z|$ and any fixed l, that

$$\sum_{l\geqslant3}\frac{(1-e^{-i\theta})^l(e^{klz}-1)}{l^2z} = O(|z|^{-1}|\theta|^3);$$

consequently, we obtain

$$\log R_k\big(z, e^{i\theta}\big) = \frac{(e^{kz}-1)}{z}(1-e^{-i\theta}) + \frac{e^{2kz}-1}{4z}(1-e^{-i\theta})^2 + O\big(|\theta| + |z|^{-1}|\theta|^3\big), \tag{2.22}$$

from which (2.20) follows. $\square$

For the use of saddle-point method, we also need asymptotic approximations to higher derivatives of $\log E_k(z)$ and $\log R_k(z,v)$ (with respect to z).

Corollary 8. *For $m \in \mathbb{Z}^+$,*

(2.23)

$$z^m \partial_z^m \log E_k(z) = z^m \partial_z^{m-1}\Big(\frac{2I(kz)}{z^2} + \frac{kz-1+e^{-kz}}{z(1-e^{-kz})}\Big) + kz\cdot 1_{m=1} + O\big(k^{-1}+|z|\big),$$

$$z^m \partial_z^m \log R_k(z, e^{i\theta}) = z^m \partial_z^m\Big(\frac{(e^{kz}-1)i\theta}{z} - \frac{(e^{kz}-1)^2\theta^2}{4z}\Big) + O\big(|\theta| + |z|^{-1}\theta^3\big),$$

uniformly as $k\to\infty$, $|z| = o(1)$, $k|z| \leqslant 2\pi-\varepsilon$ and $\theta = o(|z|^{\frac{1}{3}})$, where $\partial_z^m := \frac{\partial^m}{\partial z^m}$.

Proof. The asymptotic expansion of an analytic function can be differentiated term by term (see [29, Sec. I.2]), which then gives an expansion for the derivative. Taking derivatives with respect to z on both sides of the two expansions (2.19) and (2.20), we obtain (2.23). $\square$

Corollary 9. *Uniformly for $|t|, |\theta| = o(1)$,*

$$\begin{aligned}\log\big|R_k\big(re^{it}, e^{i\theta}\big)\big| = &-\frac{(e^{kr}-1)^2}{4r}\theta^2 - r\partial_r\Big(\frac{e^{kr}-1}{r}\Big)\theta t \\ &+ O\Big(\theta^2 + |t\theta| + \frac{\theta^4+\theta^2t^2+|t||\theta|^3)}{r}\Big).\end{aligned} \tag{2.24}$$

Proof. Since $\log\big|R_k\big(re^{it}, e^{i\theta}\big)\big| = \Re(\log R_k(re^{it}, e^{i\theta}))$, (2.24) follows from taking the real part of (2.20). $\square$

2.5. **Asymptotic negligibility of T_1.** From now on (until the end of the proof of Proposition 6), we assume $\theta = O(n^{-\frac{1}{2}})$ unless otherwise specified.

In this section, we show that T_1 (defined in (2.16)) is asymptotically negligible. Let $r > 0$ be the solution of the (approximate) saddle-point equation $rE_k'(r) = nE_k(r)$, or

$$n = \sum_{1\leqslant j\leqslant k} \frac{2jr}{1-e^{-jr}} + (k+1)r. \tag{2.25}$$

The solution exists when $0 \leqslant k \leqslant \lceil \frac{1}{2}n\rceil - 1$.

Lemma 10. *If $0 \leqslant k \leqslant \lceil \frac{1}{2}n\rceil - 1$, then the positive solution r of the equation $rE_k'(r) = nE_k(r)$ satisfies*

$$r \asymp \frac{n-2k}{(k+1)^2}. \tag{2.26}$$

In particular, r remains bounded away from zero and infinity for $k \asymp n$.

Proof. By the Euler-Maclaurin formula (see [19, Corollary 8]),

$$n - (k+1)r = \sum_{1\leqslant j\leqslant k} \frac{2jr}{1-e^{-jr}} = \frac{2I(kr)}{r} - 1 + \frac{kr}{1-e^{-kr}} + O(k^{-1}+r), \tag{2.27}$$

which holds uniformly as $k \to \infty$ and $r > 0$, where $I(z)$ is defined in (2.18). Now

$$I(x) = \begin{cases} x + \frac{1}{4}x^2 + O(x^3), \text{as } x \to 0, \\ \frac{1}{2}x^2 + \frac{\pi^2}{6} + O\big(xe^{-x}\big), \text{as } x \to \infty. \end{cases}$$

It follows that if $kr \to \infty$ (occurring when $k = o(n)$), then the RHS of (2.27) is asymptotic to $k^2 r + kr$, implying that $r \sim \frac{n}{k(k+1)}$, while if $kr \to 0$ (occurring when $k \to \frac{1}{2}n^-$), then the RHS of (2.27) is asymptotic to $2k + \frac{1}{2}k^2 r$, giving $r \sim \frac{2(n-2k)}{k^2}$. In both cases, we see that (2.26) holds. $\square$

Proposition 11. *With $k_\pm := qn \pm \sqrt{2}\varsigma n^{\frac{5}{6}}$, where ς^2 is given in (2.17), we have*

$$T_1 = \Bigg(\sum_{k\leqslant k_-} + \sum_{k\geqslant k_+}\Bigg)[z^n]E_k(z)R_k(z,e^{i\theta}) = O\big(n!\mu^n n^{\frac{5}{2}} e^{-n^{\frac{2}{3}}(1+o(1))}\big), \tag{2.28}$$

uniformly for $\theta = O(n^{-\frac{1}{2}})$.

Proof. We first exclude the extreme case when $rE_k'(r) = nE_k(r)$ has no positive solution; this occurs when n is even and $k = \frac{1}{2}n$, because (2.25) has no positive solution of r when the LHS of (2.25) equals $2k$ but the RHS is greater than $2k$. In this case, we have $[z^{2m}]E_m(z)R_m(z,v) = m!^2 v^{m+1}$, which is also bounded above by the RHS of (2.28) when $|v| = 1$.

By (2.12) and the non-negativity of $[z^n]E_k(z)$ (which implies that $|E_k(z)| \leqslant E_k(|z|)$), we have

$$\begin{aligned}[z^n]E_k(z)R_k(z,e^{i\theta}) &= O\bigg(r^{-n}\int_{-\pi}^{\pi} |E_k(re^{it})R_k(re^{it},e^{i\theta})|\,\mathrm{d}t\bigg) \\ &= O\big(r^{-n}E_k(r)e^{O(n|\theta|)}\big),\end{aligned} \tag{2.29}$$

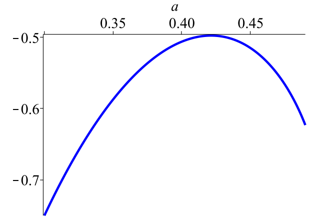
when $r \asymp n^{-1}$. Since k, r satisfy (2.27), we substitute $k = \alpha n$ and $r = \frac{\xi}{n}$ ($\xi \asymp 1$) on both sides of (2.27) and then obtain the relation $2I(\alpha\xi) = \xi$. Substituting the same k and r into (2.19) (with z replaced by r) gives

$$\begin{aligned}\log(r^{-n}E_k(r)) &= -n\log\frac{\xi}{n} + 2\alpha n \log(e^{\alpha\xi}-1) - \frac{2n}{\xi}I(\alpha\xi) + 2\log\frac{e^{\alpha\xi}-1}{\xi/n} + O(1) \\ &= \phi(\alpha,\xi)n + (n+2)\log n + O(1),\end{aligned} \tag{2.30}$$

where (using $2I(\alpha\xi) = \xi$ by (2.27))

$$\phi(\alpha,\xi) := 2\alpha\log(e^{\alpha\xi}-1) - 1 - \log\xi. \tag{2.31}$$

The maximum value of $\phi(\alpha,\xi)$ (conditioned on $2I(\alpha\xi) = \xi$; depicted on the right figure) is reached at $(\alpha,\xi) = (q,\mu^{-1})$ with μ given in (1.5) and $q = \mu\log 2$. Then we obtain $\phi(q,\mu^{-1}) = \log\mu - 1$. In addition, the concavity of $\phi(\alpha,\xi)$ when viewed as a function of α is proved in [19, Lemma 11]. We then obtain, by (2.30),



A plot of $\phi(\alpha,\xi)+1$ ($\log\mu \approx -0.4977$).

$$[z^n]E_k(z) \leqslant r^{-n}E_k(r) = O(n^{n+2}e^{\phi(q,\mu^{-1})}) = O(n^{n+2}\mu^n e^{-n}),$$

for $0 \leqslant k \leqslant \lfloor\frac{1}{2}n\rfloor - 1$.

We next improve on the growth order of $r^{-k}E_k(r)$ when $k \sim qn$. Write

$$k = qn + \varsigma\sqrt{n}x \quad \text{with} \quad x = o(\sqrt{n}). \tag{2.32}$$

Solving the equation $2I(q\xi) = \xi$ for ξ gives the expansion $r = \xi n^{-1}$, where

$$\xi = \frac{1}{\mu} - \frac{\log 2}{\varsigma\sqrt{n}}x + \frac{4\pi^2\varsigma^2(3+2\pi^2\varsigma^2) - 9(1-\log 2)}{48\pi^2\varsigma^4 n}x^2 + O\big(|x|^3 n^{-\frac{3}{2}}\big). \tag{2.33}$$

With these expansions, we then obtain ($\alpha = kn^{-1}$)

$$\phi(\alpha,\xi) = -1 + \log\mu - \frac{x^2}{2n} + \frac{\pi^4 - 72\pi^2\log 2 + 1152(\log 2)^3}{16\pi^6\varsigma^3 n^{3/2}}x^3 + O\big(x^4 n^{-2}\big).$$

This implies that with $k = qn + \varsigma\sqrt{n}x$, we have

$$r^{-n}E_k(r) = O\big(n^{n+2}\mu^n e^{-n-\frac{1}{2}x^2+O(|x|^3 n^{-\frac{1}{2}})}\big) = O\big(n!n^{\frac{3}{2}}\mu^n e^{-\frac{1}{2}x^2(1+o(1))}\big), \tag{2.34}$$

uniformly for $x = o(\sqrt{n})$. Combining this with (2.29), we then have

$$|[z^n]E_k(z)R_k(z,e^{i\theta})| = O\big(n!n^{\frac{3}{2}}\mu^n e^{-\frac{1}{2}x^2(1+o(1))+O(n|\theta|)}\big),$$

uniformly for $x = o(\sqrt{n})$. Thus, with the choice $x = \sqrt{2}n^{\frac{1}{3}}$ and by the monotonicity of $\alpha \mapsto \phi(\alpha,\xi)$, we obtain

$$T_1 = O\bigg(n\Big(\max_{k\leqslant k_-} + \max_{k\geqslant k_+}\Big)|[z^n]E_k(z)R_k(z,e^{i\theta})|\bigg) = O\big(n!n^{\frac{5}{2}}\mu^n e^{-\frac{1}{2}x^2(1+o(1))+O(n|\theta|)}\big),$$

which then implies (2.28). □

2.6. Asymptotic negligibility of T_2.

Proposition 12. *Let $k^{\pm} := qn \pm \sqrt{2}\varsigma n^{\frac{5}{8}}$. Then T_2 is bounded above by*

$$T_2 = \bigg(\sum_{k_-<k<k^-} + \sum_{k^+<k<k_+}\bigg)[z^n]E_k(z)R_k(z,e^{i\theta}) = O\big(n!n^2\mu^n e^{-n^{\frac{1}{4}}(1+o(1))}\big), \tag{2.35}$$

uniformly for $\theta = O(n^{-\frac{1}{2}})$.

Proof. Write $k = qn + \varsigma\sqrt{n}x$, where $\sqrt{2}n^{\frac{1}{8}} \leqslant |x| \leqslant \sqrt{2}n^{\frac{1}{3}}$. By Cauchy's integral representation, we have, for $k_- < k < k_+$,

$$|[z^n]E_k(z)R_k(z,e^{i\theta})| \leqslant \frac{1}{2\pi}\bigg(\int_{-t_1}^{t_1} + \int_{t_1\leqslant|t|\leqslant\pi}\bigg)r^{-n}|E_k(re^{it})R_k(re^{it},e^{i\theta})|\,\mathrm{d}t =: T_2^{[1]} + T_2^{[2]},$$

say, where $t_1 = 6n^{-\frac{1}{6}}$. For the second integral $T_2^{[2]}$, we apply the following concentration inequality ([19, Lemma 13])

$$|E_k(re^{it})| \leqslant E_k(r)\exp\Big(-\frac{k(k+1)rt^2}{\pi^2}\Big). \tag{2.36}$$

uniformly for $k \geqslant 0$, $r > 0$ and $|t| \leqslant \pi$, which then gives

$$T_2^{[2]} = \frac{r^{-n}}{2\pi}\int_{t_1\leqslant|t|\leqslant\pi}|E_k(re^{it})R_k(re^{it},e^{i\theta})|\,\mathrm{d}t = O\bigg(r^{-n}E_k(r)e^{O(n|\theta|)}\int_{t_1}^{\pi}e^{-\frac{k(k+1)rt^2}{\pi^2}}\,\mathrm{d}t\bigg).$$

Now, for $K, t_* > 0$ satisfying $\sqrt{K}\, t_* \to \infty$, we have

$$\int_{t_*}^{\infty} e^{-Kt^2}\,\mathrm{d}t = O\big((Kt_*)^{-1} e^{-Kt_*^2}\big), \tag{2.37}$$

which is proved by a direct change of variables $u = Kt^2$, and the relation $\int_t^\infty e^{-u} u^{-\frac{1}{2}}\,\mathrm{d}u \sim e^{-t} t^{-\frac{1}{2}}$ as t tends to infinity. Applying (2.37) yields

$$\begin{aligned} T_2^{[2]} &= O\Big(r^{1-n} E_k(r) t_1^{-1} e^{-\frac{k^2 r t_1^2}{\pi^2} + O(n|\theta| + t_1^2)}\Big) \\ &= O\Big(n^{-\frac{5}{6}} r^{-n} E_k(r) e^{-\frac{6}{\pi^4}(\log 2)^2 n t_1^2 + O(n|\theta| + t_1^2)}\Big), \end{aligned}$$

which, by (2.34), implies that

$$T_2^{[2]} = O\Big(n^{\frac{2}{3}} n! \mu^n e^{-\frac{1}{2}x^2 - \frac{6}{\pi^4}(\log 2)^2 n t_1^2 + O(n|\theta| + t_1^2)}\Big). \tag{2.38}$$

Since $\frac{6^3(\log 2)^2}{\pi^4} \approx 1.065 > 1$, we have

$$T_2^{[2]} = O\Big(n! \mu^n e^{-\frac{1}{2}x^2(1+o(1)) - n^{\frac{2}{3}}(1+o(1))}\Big).$$

So far we used only the simple bound (2.12) for R_k. For the remaining range in t and in k, we need the finer expansion (2.24), which together with (2.36) gives

$$\begin{aligned} \int_{-t_1}^{t_1} |R_k(re^{it}, e^{i\theta})| e^{-\frac{k(k+1)rt^2}{\pi^2}}\,\mathrm{d}t &= O\bigg(e^{-\frac{(e^{kr}-1)^2}{4r}\theta^2} \int_{-\infty}^{\infty} e^{-r\partial_r\left(\frac{e^{kr}-1}{r}\right)\theta t - \frac{k(k+1)rt^2}{\pi^2}}\,\mathrm{d}t\bigg) \\ &= O\bigg(k^{-1} r^{-\frac{1}{2}} e^{-\frac{(e^{kr}-1)^2}{4r}\theta^2 + \frac{\pi^2(e^{kr}(kr-1)+1)^2}{4k(k+1)r^3}\theta^2}\bigg) \\ &= O\big(n^{-\frac{1}{2}} e^{O(\theta^2 n)}\big), \end{aligned}$$

because

$$\int_{-\infty}^{\infty} e^{-pt - qt^2}\,\mathrm{d}t = \sqrt{\pi}\, q^{-\frac{1}{2}} e^{\frac{p^2}{4q}} \qquad (q > 0,\ p \in \mathbb{R}),$$

and $\frac{(e^{kr}-1)^2}{4r}, \frac{\pi^2(e^{kr}(kr-1)+1)^2}{4k(k+1)r^3} \asymp \frac{1}{r} \asymp n$. Consequently, by (2.34),

$$T_2^{[1]} = \frac{r^{-n}}{2\pi} \int_{-t_1}^{t_1} |E_k(re^{it}) R_k(re^{it}, e^{i\theta})|\,\mathrm{d}t = O\Big(n! n \mu^n e^{-\frac{1}{2}x^2 + O(|x|^3 n^{-\frac{1}{2}})}\Big).$$

Thus, for $|x| \geqslant \sqrt{2}\, n^{\frac{1}{8}}$,

$$|T_2| \leqslant \bigg(\sum_{k_- < k < k^-} + \sum_{k^+ < k < k_+}\bigg)(T_2^{[1]} + T_2^{[2]}) = O\Big(n! n^2 \mu^n e^{-\frac{1}{2}x^2(1+o(1))}\Big) = O\Big(n! n^2 \mu^n e^{-n^{\frac{1}{4}}(1+o(1))}\Big).$$

This proves the proposition. $\square$

2.7. **The asymptotic equivalent and the proof of Proposition 6.** We complete the proof of Proposition 6 in this subsection. Recall that $k^{\pm} := qn \pm \sqrt{2}\varsigma n^{\frac{5}{8}}$. In what follows, write $v = e^{\frac{i\vartheta}{\sqrt{n}}}$ and

$$k = qn + \varsigma\sqrt{n}x, \qquad (|x| \leqslant \sqrt{2}\, n^{\frac{1}{8}}). \tag{2.39}$$

Proposition 13. *The sum T_3 satisfies*

$$T_3 = \sum_{k^- \leqslant k \leqslant k^+} [z^n] E_k(z) R_k\big(z, e^{\frac{i\vartheta}{\sqrt{n}}}\big) = c\sqrt{n}\mu^n n! e^{\mu\sqrt{n} i\vartheta - \frac{1}{2}\vartheta^2\sigma^2}\Big(1 + O\Big(\frac{|\vartheta| + |\vartheta|^3}{\sqrt{n}}\Big)\Big), \tag{2.40}$$

uniformly for $|\vartheta| = O(1)$, where $c = \frac{12\sqrt{3}}{\pi^{5/2}}$ as in Proposition 6 and (μ, σ^2) is defined in (1.5).

Lemma 14. *Uniformly for $k^- \leqslant k \leqslant k^+$ and $\vartheta = O(1)$,*

$$\frac{[z^n] E_k(z) R_k\big(z, e^{\frac{i\vartheta}{\sqrt{n}}}\big)}{c\varsigma^{-1}\mu^n e^{-n} n^{n+\frac{1}{2}} e^{\mu\sqrt{n}\, i\vartheta - \frac{1}{2}\sigma^2\vartheta^2}} = e^{-\frac{1}{2}\left(x + \frac{3(\pi^2 - 12\log 2) i\vartheta}{\pi^4\varsigma}\right)^2}\Big(1 + O\Big(\frac{|x| + |\vartheta| + (|x| + |\vartheta|)^3}{\sqrt{n}}\Big)\Big), \tag{2.41}$$

where $c = \frac{12\sqrt{3}}{\pi^{5/2}}$, (μ, σ^2) is defined in (1.5) and ς in (2.17).

Proof. Define

$$\Xi_m(r) := m![s^m] \log E_k(re^s) R_k\big(re^s, e^{\frac{i\vartheta}{\sqrt{n}}}\big) \qquad (m = 1, 2, \dots). \tag{2.42}$$

We split Cauchy's integral representation into three parts:

$$[z^n] E_k(z) R_k\big(z, e^{\frac{i\vartheta}{\sqrt{n}}}\big) = \frac{r^{-n}}{2\pi}\Big(\int_{|t| \leqslant t_0} + \int_{t_0 < |t| \leqslant t_1} + \int_{t_1 < |t| \leqslant \pi}\Big) e^{-int} E_k(re^{it}) R_k\big(re^{it}, e^{\frac{i\vartheta}{\sqrt{n}}}\big)\, \mathrm{d}t, \tag{2.43}$$

where $t_0 := 6n^{-\frac{3}{8}}$ and $t_1 := 6n^{-\frac{1}{6}}$. We then compute the saddle-point $r = r(\vartheta)$, which satisfies the equation

$$\Xi_1(r) = \frac{rE_k'(r)}{E_k(r)} + \frac{r\partial_r R_k(r, e^{\frac{i\vartheta}{\sqrt{n}}})}{R_k(r, e^{\frac{i\vartheta}{\sqrt{n}}})} = n. \tag{2.44}$$

Asymptotically, by (2.23),

$$\begin{aligned} &\frac{2I(kr)}{r} + \frac{e^{kr}(kr - 1) + 1}{r\sqrt{n}} i\vartheta \\ &\quad + \frac{1}{rn}\Big(\frac{kr^2 n}{e^{kr} - 1} + rn(2kr - 1) - \frac{(e^{kr} - 1)(e^{kr}(2kr - 1) + 1)\vartheta^2}{4}\Big) + O\big(n^{-\frac{1}{2}}\big) = n. \end{aligned} \tag{2.45}$$

We then deduce the expansion

$$r = r(\vartheta) = \frac{1}{\mu n} + \frac{r_1}{n^{3/2}} + \frac{r_2}{n^2} + \cdots, \tag{2.46}$$

by a standard bootstrapping (or asymptotic iteration) argument (see [10, Sec. 2.5]) using the form (2.39) and the Taylor expansion

$$I(\log 2 + t) = \frac{\pi^2}{12} + 2(\log 2)t + (1 - \log 2)t^2 + \cdots \qquad (t \sim 0).$$

Here each coefficient r_j is a polynomial of x and ϑ of degree j; in particular, $r_1 = -\frac{\log 2}{\varsigma} x - \frac{3(2\log 2 - 1)}{2\pi^2\varsigma^2} i\vartheta$, and the expressions of r_j's are messy for $j \geqslant 2$.

With such k and r, we deduce, by (2.23), that

$$\Xi_m(r) \asymp n \qquad (m = 1, 2, \ldots). \tag{2.47}$$

In particular, by (2.23) and the relation (2.44), we have

$$\Xi_2(r) = \frac{e^{kr}(2k^2 r - n) + n}{e^{kr} - 1} + \frac{k^2 r e^{kr}}{\sqrt{n}} i\vartheta + O(1),$$

which then satisfies, by substituting (2.39) and (2.46),

$$\Xi_2(r) = \tfrac{2}{3}\pi^2 \varsigma^2 n + O\big((|x| + |\vartheta|)\sqrt{n}\big). \tag{2.48}$$

Then our choice of $t_0 \asymp n^{-\frac{3}{8}}$ implies that $\Xi_2(r)t_0^2 \to \infty$ and $\Xi_m(r)t_0^m \to 0$ for $m \geqslant 3$. It follows that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{|t|\leqslant t_0} e^{-int} E_k(re^{it}) R_k\big(re^{it}, e^{\frac{i\vartheta}{\sqrt{n}}}\big)\,\mathrm{d}t \\
&= \frac{E_k(r)R_k\big(r, e^{\frac{i\vartheta}{\sqrt{n}}}\big)}{2\pi}\int_{-t_0}^{t_0} e^{-\frac{1}{2}\Xi_2(r)t^2 - \frac{1}{6}\Xi_3(r)it^3 + O(n|t|^4)}\,\mathrm{d}t \\
&= \frac{E_k(r)R_k\big(r, e^{\frac{i\vartheta}{\sqrt{n}}}\big)}{2\pi\sqrt{\Xi_2(r)}}\int_{-t_0\sqrt{\Xi_2(r)}}^{t_0\sqrt{\Xi_2(r)}} e^{-\frac{1}{2}u^2}\Big(1 - \frac{\Xi_3(r)}{6\Xi_2(r)^{3/2}} iu^3 + O\big(n^{-1}(u^4 + u^6)\big)\Big)\,\mathrm{d}u \\
&= \frac{E_k(r)R_k\big(r, e^{\frac{i\vartheta}{\sqrt{n}}}\big)}{\sqrt{2\pi\,\Xi_2(r)}}\big(1 + O\big(n^{-1}\big)\big).
\end{aligned}$$

This will prove the saddle-point approximation

$$[z^n]E_k(z)R_k\big(z, e^{\frac{i\vartheta}{\sqrt{n}}}\big) = \frac{r^{-n}E_k(r)R_k\big(r, e^{\frac{i\vartheta}{\sqrt{n}}}\big)}{\sqrt{2\pi\,\Xi_2(r)}}\Big(1 + O\big(|\Xi_2(r)|^{-1}\big)\Big), \tag{2.49}$$

uniformly for $k^- \leqslant k \leqslant k^+$, provided that the smallness of the two other integrals in (2.43) is justified.

Consider first the middle integral in (2.43). Observe that, by the definition (2.42) of $\Xi_m(r)$ and the choice of r,

$$\begin{aligned}
-int + \log\frac{E_k(re^{it})R_k\big(re^{it}, e^{\frac{i\vartheta}{\sqrt{n}}}\big)}{E_k(r)R_k\big(r, e^{\frac{i\vartheta}{\sqrt{n}}}\big)} &= -\frac{1}{2}\Xi_2(r)t^2 + \sum_{m\geqslant 3}\frac{\Xi_m(r)}{m!}(it)^m \\
&= -\frac{1}{2}\Xi_2(r)t^2(1 + o(1)),
\end{aligned}$$

uniformly for $t_0 < |t| \leqslant t_1$, because $t = o(1)$ in this range and each $\Xi_m(r)$ is linear in n (see (2.47)) so that the Taylor expansion is itself an asymptotic expansion. Then, by applying (2.37),

$$\begin{aligned}
r^{-n}\int_{t_0<|t|\leqslant t_1} e^{-int}E_k(re^{it})R_k\big(re^{it}, e^{\frac{i\vartheta}{\sqrt{n}}}\big)\,\mathrm{d}t &= O\Big(r^{-n}E_k(r)R_k\big(r, e^{\frac{i\vartheta}{\sqrt{n}}}\big)\int_{t_0}^{\infty} e^{-\frac{1}{2}\Xi_2(r)t^2(1+o(1))}\,\mathrm{d}t\Big) \\
&= O\Big(\big|r^{-n}E_k(r)R_k\big(r, e^{\frac{i\vartheta}{\sqrt{n}}}\big)\big|n^{-\frac{1}{4}}e^{-n^{\frac{1}{4}}(1+o(1))}\Big),
\end{aligned}$$

which is much smaller than the RHS of (2.49). Here we used the approximation

$$\frac{1}{2}\Xi_2(r)\, t_0^2 \sim 12\pi^2\varsigma^2 n^{\frac{1}{4}} \approx 3.029\, n^{\frac{1}{4}} > n^{\frac{1}{4}}.$$

On the other hand, again by substituting k by (2.39) and r by (2.46) in (2.19) and (2.20), we are led to

$$\begin{aligned}
&-n\log r + \log E_k(r) + \log R_k(r, e^{i\vartheta/\sqrt{n}})\\
&= (\log\mu + \log n - 1)\, n + \mu i\vartheta\sqrt{n} - \frac{1}{2}x^2 - \frac{3\vartheta(\pi^2 - 12\log 2)i}{\varsigma\pi^4}x\\
&\quad + \log(c_0 n) + \frac{-72\log^2 2 - 3\pi^2 + 144\log 2 - 36}{2\pi^2(-24\log^2 2 + \pi^2)}\vartheta^2 + O\left(\frac{|x| + |\vartheta| + (|x| + |\vartheta|)^3}{\sqrt{n}}\right),\\
&= (\log\mu + \log n - 1)\, n + \mu i\vartheta\sqrt{n} - \frac{1}{2}\left(x + \frac{3\vartheta(\pi^2 - 12\log 2)i}{\varsigma\pi^4}\right)^2\\
&\quad + \log(c_0 n) + \log c_0 - \frac{1}{2}\sigma^2\vartheta^2 + O\left(\frac{|x| + |\vartheta| + (|x| + |\vartheta|)^3}{\sqrt{n}}\right),
\end{aligned}$$

uniformly for $x = o(n^{\frac{1}{6}})$ and $\vartheta = O(1)$, where $c_0 = \frac{24}{\pi}$, σ^2 is defined in (1.5) and ς in (2.17). This is asymptotically equivalent to the expression:

$$\frac{r^{-n}E_k(r)R_k\big(r, e^{i\vartheta/\sqrt{n}}\big)}{c_0\mu^n e^{-n} n^{n+1} e^{\mu\sqrt{n}\, i\vartheta - \frac{1}{2}\sigma^2\vartheta^2}} = e^{-\frac{1}{2}\left(x + \frac{3(\pi^2 - 12\log 2)i\vartheta}{\pi^4\varsigma}\right)^2}\left(1 + O\Big(\frac{|x| + |\vartheta| + (|x| + |\vartheta|)^3}{\sqrt{n}}\Big)\right). \tag{2.50}$$

It remains to estimate the integral over the range $t_1 < |t| \leqslant \pi$ (that is the third integral in (2.43)). The estimation is similar to the analysis of T_2 except that r here is not a purely real number when $\vartheta \neq 0$. Indeed, in view of the form (2.46), we have the expansion

$$|r| = |r(\vartheta)| = \frac{1}{\mu n} + \frac{\Re(r_1)}{n^{3/2}} + \frac{2\Re(r_2) + \mu\Im(r_1)^2}{2n^2} + \cdots, \tag{2.51}$$

where $\Re(r_1) = -\frac{\log 2}{\varsigma}x$. The integral over the range $t_0 < |t| \leqslant \pi$ is estimated as in the case of T_2 (see (2.38)):

$$\begin{aligned}
\left| r^{-n}\int_{t_0 < |t| \leqslant \pi} e^{-int}E_k(re^{it})R_k\big(re^{it}, e^{\frac{i\vartheta}{\sqrt{n}}}\big)\,\mathrm{d}t\right| &= |r|^{-n}E_k(|r|)e^{O(\sqrt{n}|\vartheta|)}\int_{t_0}^{\infty} e^{-\frac{k(k+1)|r|t^2}{\pi^2}}\,\mathrm{d}t\\
&= O\big(|r|^{-n}E_k(|r|)e^{O(\sqrt{n}|\vartheta|) - n^{\frac{2}{3}}(1+o(1))}\big),
\end{aligned}$$

uniformly for $k^- \leqslant k \leqslant k^+$ and $\vartheta = O(1)$. By the expansion (2.51), we have

$$\begin{aligned}
-n\log|r| &= n\log n + n\log\mu + \frac{q}{\varsigma}x\sqrt{n} - \left(\frac{6\pi^2\varsigma^2 + 9(\log 2 - 1)}{8\pi^4\varsigma^4} + \frac{1}{2}\right)x^2 + O(1),\\
\log E_k(|r|) &= -n - \frac{q}{\varsigma}x\sqrt{n} + \frac{6\pi^2\varsigma^2 + 9(\log 2 - 1)}{8\pi^4\varsigma^4}x^2 + \log n + O(1),
\end{aligned}$$

so that

$$|r|^{-n}E_k(|r|) = O\big(n^{n+1}\mu^n e^{-n - \frac{1}{2}x^2}\big),$$

and we obtain

$$\left| r^{-n} \int_{t_0<|t|\leqslant\pi} e^{-int} E_k(re^{it}) R_k\big(re^{it}, e^{\frac{i\vartheta}{\sqrt{n}}}\big)\,\mathrm{d}t \right| = O\big(n^{n+1}\mu^n e^{-n-\frac{1}{2}x^2-n^{\frac{2}{3}}(1+o(1))}\big).$$

Note that in (2.50)

$$\left|e^{-\frac{1}{2}\left(x+\frac{3(\pi^2-12\log 2)i\vartheta}{\pi^4\varsigma}\right)^2}\right| = O(e^{-\frac{1}{2}x^2}),$$

since $\vartheta = O(1)$. Collecting these expansions, we obtain (2.49). Then (2.41) follows from (2.49) and (2.50). $\square$

We now prove Proposition 13 by summing (2.41) over $k^- \leqslant k \leqslant k^+$ and then approximating the sum by a Gaussian integral; in this way, we obtain

$$\begin{aligned}
\frac{\sum_{k^-\leqslant k\leqslant k^+}[z^n]E_k(z)R_k\big(z,e^{\frac{i\vartheta}{\sqrt{n}}}\big)}{\sqrt{2\pi}c\mu^n e^{-n}n^{n+\frac{1}{2}}e^{\mu\sqrt{n}i\vartheta-\frac{1}{2}\sigma^2\vartheta^2}} &= \sum_{k^-\leqslant k\leqslant k^+} \frac{e^{-\frac{1}{2}\left(x+\frac{3(\pi^2-12\log 2)i\vartheta}{\pi^4\varsigma}\right)^2}}{\sqrt{2\pi}\varsigma}\Big(1+O\Big(\frac{|\vartheta|+|\vartheta|^3}{\sqrt{n}}\Big)\Big)\\
&= \varsigma\sqrt{n}\int_{2n^{1/8}}^{-2n^{1/8}} \frac{e^{-\frac{1}{2}\left(x+\frac{3(\pi^2-12\log 2)i\vartheta}{\pi^4\varsigma}\right)^2}\,\mathrm{d}x}{\sqrt{2\pi}\varsigma}\Big(1+O\Big(\frac{|\vartheta|+|\vartheta|^3}{\sqrt{n}}\Big)\Big)\\
&= \sqrt{n}\Big(1+O\Big(\frac{|\vartheta|+|\vartheta|^3}{\sqrt{n}}\Big)\Big).
\end{aligned}$$

By Stirling's formula for $n!$, we obtain (2.40), which completes the proof of Proposition 13, which in turn proves Proposition 6 by collecting the estimate (2.28) for T_1 and (2.35) for T_2.

2.8. **Proof of Theorem 1.** We now translate the asymptotic approximation (2.10) for $[z^n]E(z,v)$ when $\Lambda(z) = e^z$ into that for general Λ with $\lambda_1 > 0$ (namely, $[z^n]F(z,v)$; see (2.3)) . By (2.3),

$$P_n(v) := [z^n]F(z,v) = [z^n]v\Lambda(z) \sum_{0\leqslant k\leqslant\lfloor\frac{1}{2}n\rfloor} \prod_{1\leqslant j\leqslant k} \frac{\Lambda(z)(\Lambda(z)^j-1)^2}{1-(1-v^{-1})\Lambda(z)^j}.$$

Since $\Lambda(z) = 1+\lambda_1 z+\cdots$ with $\lambda_1 \neq 0$ is analytic at $z = 0$, the function is locally invertible at $z = 0$ and we can make the change of variables $\Lambda(z) = e^y$, namely, let $z = \zeta(y)$ so that $\Lambda(\zeta(y)) = e^y$ (where $\zeta(y)$ is also analytic at $y = 0$). Then, by Cauchy's integral formula, we have, for $n \geqslant 1$,

$$\begin{aligned}
P_n(v) &= \frac{1}{2\pi i}\oint_{|z|=r_0} z^{-n-1}v\Lambda(z) \sum_{0\leqslant k\leqslant\lfloor\frac{1}{2}n\rfloor} \prod_{1\leqslant j\leqslant k} \frac{\Lambda(z)(\Lambda(z)^j-1)^2}{1-(1-v^{-1})\Lambda(z)^j}\,\mathrm{d}z\\
&= \frac{1}{2\pi i}\oint_{|y|=r} \zeta(y)^{-n-1}\zeta'(y) \sum_{0\leqslant k\leqslant\lfloor\frac{1}{2}n\rfloor} E_k(y)R_k(y,v)\,\mathrm{d}y\\
&= [y^n]h_n(y) \sum_{0\leqslant k\leqslant\lfloor\frac{1}{2}n\rfloor} E_k(y)R_k(y,v),
\end{aligned}$$

where $h_n(y) := \left(\frac{y}{\zeta(y)}\right)^{n+1}\zeta'(y)$. When $y \sim 0$, we have the Taylor expansion $\zeta(y) = \sum_{j\geqslant 1}\zeta_j y^j$, where $\zeta_1 = \frac{1}{\lambda_1}$, and $\zeta_2 = \frac{1}{\lambda_1}\left(\frac{1}{2} - \frac{\lambda_2}{\lambda_1^2}\right)$. Then, with $y = \frac{t}{n}$, we have the asymptotic expansion

$$h_n\Big(\frac{t}{n}\Big) \sim \lambda_1^n e^{-\frac{\zeta_2}{\zeta_1}t}\Big(1 + \sum_{j\geqslant 1}\upsilon_j(t)n^{-j}\Big), \tag{2.52}$$

for $t = O(1)$, where $\upsilon_j(t)$ is a polynomial in t of degree $j+1$. In particular, $\upsilon_1(t) = \frac{\zeta_2}{\zeta_1}t + \frac{\zeta_2^2 - 2\zeta_1\zeta_3}{2\zeta_1^2}t^2$. Consequently,

$$P_n(v) = \lambda_1^n n^n [t^n] e^{-\frac{\zeta_2}{\zeta_1}t}\Big(1 + O\Big(\frac{|t| + |t|^2}{n}\Big)\Big)\sum_{0\leqslant k\leqslant \lfloor \frac{1}{2}n\rfloor} E_k\Big(\frac{t}{n}\Big)R_k\Big(\frac{t}{n}, v\Big). \tag{2.53}$$

Following the proof of Proposition 6, we know that only a small neighborhood of $(k,t) \sim (qn, \frac{1}{\mu})$ contributes dominantly; furthermore, the extra factor $e^{-\frac{\zeta_2}{\zeta_1}t}$ before the sum in (2.53) is bounded. Thus we substitute $t = rn$, with r expanding as in (2.46), and obtain

$$e^{-\frac{\zeta_2}{\zeta_1}t} = e^{-\frac{\zeta_2}{\zeta_1\mu}}\big(1 + O(n^{-\frac{1}{2}})\big). \tag{2.54}$$

This, together with (2.53) and (2.10) of Proposition 6, implies that

$$\begin{aligned}\lambda_1^{-n}P_n\big(e^{\frac{i\vartheta}{\sqrt{n}}}\big) &= n^n e^{-\frac{\zeta_2}{\zeta_1\mu}}\big(1 + O\big(n^{-\frac{1}{2}}\big)\big)[t^n]E\Big(\frac{t}{n}, e^{\frac{i\vartheta}{\sqrt{n}}}\Big) + O\Big(n^{n-1}\Big|[t^{n-1}]E\Big(\frac{t}{n}, e^{\frac{i\vartheta}{\sqrt{n}}}\Big)\Big|\Big)\\ &= c\sqrt{n}\mu^n n! e^{\mu\sqrt{n}i\vartheta - \frac{1}{2}\sigma^2\vartheta^2}\big(1 + O\big((1 + |\vartheta| + |\vartheta|^3)n^{-\frac{1}{2}}\big)\big),\end{aligned}$$

where $c := \frac{12\sqrt{3}}{\pi^{5/2}}e^{\frac{\pi^2}{6}\left(\frac{\lambda_2}{\lambda_1^2} - \frac{1}{2}\right)}$. Then, as in (2.11),

$$\mathbb{E}\big(e^{\frac{X_n - \mu n}{\sigma\sqrt{n}}i\vartheta}\big) = \frac{P_n\big(e^{\frac{i\vartheta}{\sigma\sqrt{n}}}\big)}{P_n(1)}e^{-\frac{\mu}{\sigma}\sqrt{n}i\vartheta} = e^{-\frac{1}{2}\vartheta^2}\big(1 + O\big((1 + |\vartheta| + |\vartheta|^3)n^{-\frac{1}{2}}\big)\big),$$

which then implies Theorem 1 by the continuity theorem for characteristic functions; see [4, Theorem 26.3].

3. The mean and the variance of the dimension

We prove Theorem 2 in this section, together with a few related properties.

3.1. **The generating functions of moments.**

Recall that a_n is defined in (1.3). Define

$$M_h(z) := \sum_{n\geqslant 0} a_n\mathbb{E}(X_n^h)z^n = \partial_s^h F(z, e^s)|_{s=0} \qquad (h = 0, 1, \dots) \tag{3.1}$$

to be (up to the normalizing factor a_n) the generating function of the hth moment of X_n, where F is given in (1.4). In particular, $M_0(z)$ equals the generating function in (1.3).

Lemma 15. *The generating function of the hth moment of X_n satisfies*

$$M_h(z) = U_h(z)M_0(z) + V_h(z), \tag{3.2}$$

for $h \geqslant 1$ with $M_h(0) = 0$, where ($\left\{ {h \atop j} \right\}$ are the Stirling numbers of the second kind)

$$U_h(z) := \sum_{0 \leqslant \ell \leqslant h} \left\{ {h+1 \atop \ell+1} \right\} (-1)^{h-\ell} \ell! \prod_{1 \leqslant j \leqslant \ell} \frac{1}{1 - \Lambda(z)^{-j}} \tag{3.3}$$

$$V_h(z) := \sum_{0 \leqslant \ell \leqslant h} \left\{ {h+1 \atop \ell+1} \right\} (-1)^{h+1-\ell} \ell! \sum_{0 \leqslant k < \ell} \prod_{k < j \leqslant \ell} \frac{1}{1 - \Lambda(z)^{-j}}.$$

Proof. By taking the derivative with respect to v on both sides of (1.4) and then substituting $v = 1$, we obtain

$$\begin{aligned} M_1(z) = \partial_v F(z,v)\big|_{v=1} &= \sum_{k \geqslant 0} \Big(\sum_{1 \leqslant l \leqslant k} \Lambda(z)^{-l} \Big) \Big(\prod_{1 \leqslant j \leqslant k} \big(1 - \Lambda(z)^{-j}\big) \Big) \\ &= \sum_{k \geqslant 0} \Big(-1 + \frac{1 - \Lambda(z)^{-k-1}}{1 - \Lambda(z)^{-1}} \Big) \Big(\prod_{1 \leqslant j \leqslant k} \big(1 - \Lambda(z)^{-j}\big) \Big). \end{aligned}$$

It follows that

$$M_1(z) = \frac{M_0(z) - \Lambda(z)}{\Lambda(z) - 1}. \tag{3.4}$$

In a similar way,

$$\begin{aligned} M_2(z) &= \big(\partial_v^2 F(z,v) + \partial_v F(z,v)\big)\big|_{v=1} \\ &= \sum_{k \geqslant 0} \Big(1 - 3 \frac{1 - \Lambda(z)^{-k-1}}{1 - \Lambda(z)^{-1}} + 2 \frac{\big(1 - \Lambda(z)^{-k-1}\big)\big(1 - \Lambda(z)^{-k-2}\big)}{\big(1 - \Lambda(z)^{-1}\big)\big(1 - \Lambda(z)^{-2}\big)} \Big) \Big(\prod_{1 \leqslant j \leqslant k} \big(1 - \Lambda(z)^{-j}\big) \Big). \end{aligned}$$

Thus

$$\begin{aligned} M_2(z) &= M_0(z) - \frac{3\big(M_0(z) - 1\big)}{1 - \Lambda(z)^{-1}} + \frac{2\big(M_0(z) - 2 + \Lambda(z)^{-1}\big)}{\big(1 - \Lambda(z)^{-1}\big)\big(1 - \Lambda(z)^{-2}\big)} \\ &= \frac{1 + 2\Lambda(z) - \Lambda(z)^2}{\big(\Lambda(z) - 1\big)\big(\Lambda(z)^2 - 1\big)} M_0(z) - \frac{\Lambda(z)\big(3 - 2\Lambda(z) + \Lambda(z)^2\big)}{\big(\Lambda(z) - 1\big)\big(\Lambda(z)^2 - 1\big)}. \end{aligned} \tag{3.5}$$

The general form (3.2) is then proved by the same arguments and induction. ☐

3.2. **Combinatorial interpretations.** From (3.4), we have the identity

$$\sum_{1 \leqslant j < n} \lambda_j a_{n-j} \mu_{n-j} = a_n - \lambda_n \qquad (n \geqslant 1),$$

where $\mu_n = \mathbb{E}(X_n)$ (see (1.6)) and $a_n \mu_n = [z^n] M_1(z)$. In particular, in the primitive case when $\Lambda(z) = 1 + z$, we have the surprisingly simple identity (1.6) for the expected dimension, which says that *the expected dimension of a random primitive FM of size n equals the ratio of the number of primitive FMs of size $n+1$ and that of size n.*

Similarly, for the second moment, we have the identity

$$a_n \mathbb{E}\big(X_n^2\big) + 2a_{n+1} \mathbb{E}\big(X_{n+1}^2\big) = 2a_{n+3} - a_{n+1}.$$

These simple relations certainly demand for combinatorial interpretations, which are given in the following forms.

Proposition 16. *Let $p_{n,k}$ denote the number of primitive FMs of size n and dimension k, and let a_n be the number of primitive FMs of size n. Then for $n \geqslant 1$,*

$$a_{n+1} = \sum_{1 \leqslant k \leqslant n} k p_{n,k}, \tag{3.6}$$

$$a_{n+3} = \sum_{1 \leqslant k \leqslant n} \binom{k+1}{2} p_{n,k} + \sum_{1 \leqslant k \leqslant n+1} k^2 p_{n+1,k}. \tag{3.7}$$

Among the diverse Fishburn structures, we find it simpler to interpret (3.6) and (3.7) in the language of ascent sequences, listed in Table 1. We can then translate the recursive construction on primitive ascent sequences into primitive FMs via the bijection in [11].

Definition 1 (Ascent sequence)**.** Let $\mathcal{I}_n$ be the set of inversion sequences of length n, namely,

$$\mathcal{I}_n := \{s = (s_1, s_2, \ldots, s_n) : 0 \leqslant s_j < j, 1 \leqslant j \leqslant n\},$$

For any sequence $s \in \mathcal{I}_n$, let

$$\mathsf{asc}(s) := |\{1 \leqslant j < n : s_j < s_{j+1}\}| \tag{3.8}$$

be the number of ascents of s. An *inversion sequence* $s \in \mathcal{I}_n$ is an *ascent sequence* if for all $2 \leqslant j \leqslant n$, s_j satisfies $s_j \leqslant \mathsf{asc}(s_1, s_2, \ldots, s_{j-1}) + 1$. An ascent sequence is *primitive* if no consecutive entries are identical.

Proof. (Proposition 16) It is known (see [6, 11]) that $p_{n,k}$ also enumerates the number of primitive ascent sequences of length n with $k-1$ ascents. For instance, $p_{4,3} = 4$: the corresponding primitive ascent sequences are 0121, 0120, 0102 and 0101 and they are in bijection with the following primitive FMs from left to right, respectively.

$$\begin{pmatrix} 1 & 0 & 0 \\ & 1 & 1 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ & 0 & 1 \\ & & 1 \end{pmatrix}$$

Given a primitive ascent sequence s of length n and with $k-1$ ascents, we add a new entry at the end of s, which can be any integer from $[0, k]$ but not equal to the last entry s_n of s. In other words, there are k possible ways to add such an integer so that the resulting sequence is a primitive ascent sequence of length $n+1$, which leads to (3.6).

Now we extend the same proof to show (3.7). Given a primitive ascent sequence s of length $n+1$ and with $k-1$ ascents, we add two entries x, y at the end of s, such that the resulting sequence is a primitive ascent sequence and if x (the penultimate entry) is removed, it is still an ascent sequence (not necessarily primitive). According to Definition 1, we have $0 \leqslant x \leqslant k$, $x \neq s_{n+1}$, and $0 \leqslant y \leqslant k$, $y \neq x$. That is, there are k^2 possible values for the pair (x, y) so that we obtain an primitive ascent sequence of length $n+3$ and if the penultimate entry is removed, it is still an ascent sequence.

On the other hand, given a primitive ascent sequence s of length n and with $k-1$ ascents, we add three entries x, y, z at the end of s so that the resulting sequence is a primitive ascent sequence but if y (the penultimate entry) is removed, then it is no longer an ascent sequence. According to these conditions, we have $x < y < z = \mathsf{asc}(s_1 \cdots s_n x) + 2$, implying that z is determined by x and the sequence s. It remains to count the number of pairs (x, y) for which $s_1 \cdots s_n x y$ is a primitive ascent sequence and $x < y$. If $y = k+1$, then $s_n < x \leqslant k$, that is, there are $k - s_n$ possible choices for x; otherwise $0 \leqslant y \leqslant k$. Since $0 \leqslant x < y \leqslant k$ and $x \neq s_n$, there are $\frac{1}{2}k(k-1)+s_n$ different values for the pair (x, y). It follows that there are in total $\frac{1}{2}k(k-1)+k=\frac{1}{2}k(k+1)$ choices for (x, y) so that $s_1 \cdots s_n x y$ is a primitive ascent sequence with $x < y$.

Since any primitive ascent sequence of length $n+3$ can be produced by either construction, we thus conclude the identity (3.7). □

Remark 3. *When $\Lambda(z)=(1-z)^{-1}$, that is, a_n denotes the number of FMs of size n, we have instead the pair of relations ($\bar{p}_{n,k}$ denoting the number of size-n FMs of dimension k, which is [25, A137251]):*

$$\begin{cases} a_{n+1}-a_n = \displaystyle\sum_{1\leqslant k\leqslant n} k\,\bar{p}_{n,k}, \\ a_{n+3}-a_{n+2} = \displaystyle\sum_{1\leqslant k\leqslant n+1} k(k+2)\,\bar{p}_{n+1,k} - \sum_{1\leqslant k\leqslant n}\binom{k+1}{2}\bar{p}_{n,k}. \end{cases}$$

In words: The expected dimension of a random FM of size n equals the ratio of a_{n+1} and a_n minus 1. *Similar combinatorial interpretations can be given as in the primitive case.*

3.3. **Asymptotics of the moments.** We first recall a refined expansion of (1.3), which is helpful in deriving the asymptotics of the first two moments and proving Theorem 2.

Proposition 17 (A refinement of Corollary 20 from [19]). *Assume that $\Lambda(z)$ is analytic at $z=0$ and $\lambda_1 > 0$. Then the number of Λ-FMs of size n satisfies the asymptotic expansion*

$$\frac{a_n}{c n^{\frac{1}{2}}(\lambda_1\mu)^n n!} = 1 + \sum_{1\leqslant j<J} d_j n^{-j} + O\big(n^{-J}\big), \tag{3.9}$$

for any positive integer J, where (c, μ) is as in (1.3), and, in particular,

$$\begin{aligned} d_1 &= \frac{3}{8} + \frac{19\lambda_1^2-36\lambda_2}{144\lambda_1^2}\pi^2 + \frac{\lambda_1^2+12\lambda_1\lambda_3-12\lambda_2^2}{432\lambda_1^4}\pi^4, \\ d_2 &= -\frac{7}{128} - \frac{(19\lambda_1^2-36\lambda_2)\pi^2}{1152\lambda_1^2} - \frac{(35\lambda_1^4+456\lambda_1^2\lambda_2+1872\lambda_1\lambda_3-2304\lambda_2^2)\pi^4}{41472\lambda_1^4} \\ &\quad + \frac{(7\lambda_1^6-12\lambda_1^4\lambda_2+228\lambda_1^3\lambda_3-228\lambda_1^2\lambda_2^2+288\lambda_1^2\lambda_4-1008\lambda_1\lambda_2\lambda_3+720\lambda_2^3)\pi^6}{62208\lambda_1^6} \\ &\quad - \frac{(5\lambda_1^4-12\lambda_1^2\lambda_2+24\lambda_1\lambda_3-12\lambda_2^2)(\lambda_1^4-12\lambda_2\lambda_1^2-24\lambda_1\lambda_3+36\lambda_2^2)\pi^8}{1492992\lambda_1^8}. \end{aligned} \tag{3.10}$$

This is derived by refining the saddle-point analysis on the generating functions $E(z, 1)$ and $F(z, 1)$ in the previous section. We omit the details as all steps can be coded in symbolic computation software. See [19, Section 4.1] for an alternative approach to (3.9), based on Zagier's approach (which in turn relies on other identities and quantum modular forms).

We list the expressions of d_1 and d_2 in the two standard cases of FMs:

	$\Lambda(z)=(1-z)^{-1}$	$\Lambda(z)=1+z$
d_1	$\frac{3}{8}-\frac{17\pi^2}{144}+\frac{\pi^4}{432}$	$\frac{3}{8}+\frac{19\pi^2}{144}+\frac{\pi^4}{432}$
d_2	$-\frac{7}{128}+\frac{17\pi^2}{1152}-\frac{59\pi^4}{41472}-\frac{5\pi^6}{62208}-\frac{5\pi^8}{1492992}$	$-\frac{7}{128}-\frac{19\pi^2}{1152}-\frac{35\pi^4}{41472}+\frac{7\pi^6}{62208}-\frac{5\pi^8}{1492992}$

In particular, the expression d_1 is consistent with the expression given in [30, p. 955].

Now we are ready to prove Theorem 2. Write first (3.4) as

$$M_1(z) = \frac{M_0(z)-1}{\Lambda(z)-1} - 1 = \frac{M_0(z)-1}{z}\cdot\frac{z}{\Lambda(z)-1} - 1.$$

We then have, for $n \geqslant 1$,

$$[z^n]M_1(z) = \sum_{0\leqslant j\leqslant n} a_{n+1-j}\cdot[z^j]\frac{z}{\Lambda(z)-1}. \tag{3.11}$$

By the analyticity of $\frac{z}{\Lambda(z)-1}$ at the origin (because $\Lambda(z)$ is also analytic and $\lambda_1 > 0$), we see that the coefficient $[z^n]\frac{z}{\Lambda(z)-1}$ *is of exponential order* (or equivalently its logarithm is of order $O(n)$; see definitions in [15, Theorem IV.7, page 244]). Since a_n is of *factorial order* (or equivalently $\log a_n \sim n\log n$) for large n (see (3.9)), we deduce that the partial sum (3.11) is itself an asymptotic expansion in the following sense

$$\mathbb{E}(X_n) = \frac{[z^n]M_1(z)}{a_n} = \sum_{0\leqslant j\leqslant J}\frac{a_{n+1-j}}{a_n}\cdot[z^j]\frac{z}{\Lambda(z)-1} + O\big(n^{-J}\big),$$

for any nonnegative integer J; see [3, Theorem 2] or [15, Theorem VI.12, page 434] for a general theorem. In particular,

$$\mathbb{E}(X_n) = \frac{a_{n+1}}{\lambda_1 a_n} - \frac{\lambda_2}{\lambda_1^2} + O\Big(\frac{a_{n-1}}{a_n}\Big),$$

which, together with (3.9), gives

$$\mathbb{E}(X_n) = \frac{a_{n+1}}{\lambda_1 a_n} - \frac{\lambda_2}{\lambda_1^2} + O\big(n^{-1}\big) = \mu\Big(n+\frac{3}{2}\Big) - \frac{\lambda_2}{\lambda_1^2} + O\big(n^{-1}\big).$$

This proves (1.7), the first part of Theorem 2. Note that with the weaker form (1.3), the constant term cannot be made explicit. Further terms can be readily computed by computer algebra software; for example, using the expression of d_1 in (3.10),

$$\mathbb{E}(X_n) = \mu\Big(n+\frac{3}{2}\Big) - \frac{\lambda_2}{\lambda_1^2} + \frac{1}{n}\Big(\frac{1}{2\pi^2} - \frac{19}{24} - \frac{\pi^2}{72\lambda_1^2} + \frac{3\lambda_2}{2\lambda_1^2} - \frac{2\pi^2\lambda_3}{3\lambda_1^3} + \frac{2\pi^2\lambda_2^2}{3\lambda_1^4}\Big) + O\big(n^{-2}\big).$$

Similarly, by writing (3.5) in the form

$$M_2(z) = \frac{1+2\Lambda(z)-\Lambda(z)^2}{(\Lambda(z)+1)(\Lambda(z)-1)^2}\,(M_0(z)-1) - \frac{\Lambda(z)^2+1}{\Lambda(z)^2-1},$$

we have

$$\mathbb{E}\big(X_n^2\big) = \sum_{0\leqslant j\leqslant n+1}\frac{a_{n+2-j}}{a_n}\cdot[z^j]\frac{z^2(1+2\Lambda(z)-\Lambda(z)^2)}{(\Lambda(z)+1)(\Lambda(z)-1)^2} - \frac{1}{a_n}[z^n]\frac{\Lambda(z)^2+1}{\Lambda(z)^2-1}.$$

By a similar argument used above, we then deduce that

$$\mathbb{E}\big(X_n^2\big) = \frac{1}{\lambda_1^2}\cdot\frac{a_{n+2}}{a_n} - \frac{\lambda_1^2+4\lambda_2}{2\lambda_1^3}\cdot\frac{a_{n+1}}{a_n} - \frac{\lambda_1^4 - 2\lambda_1^2\lambda_2 + 8\lambda_1\lambda_3 - 12\lambda_2^2}{4\lambda_1^4} + O\big(n^{-1}\big),$$

and accordingly

$$
\begin{aligned}
\mathbb{V}(X_n) &= \frac{[z^n]M_2(z)}{a_n} - \mu_n^2 \\
&= \frac{1}{\lambda_1^2}\cdot\frac{a_{n+2}}{a_n} - \frac{(\lambda_1^2+4\lambda_2)}{2\lambda_1^3}\cdot\frac{a_{n+1}}{a_n} - \frac{\lambda_1^4 - 2\lambda_1^2\lambda_2 + 8\lambda_1\lambda_3 - 12\lambda_2^2}{4\lambda_1^4} + O\Big(\frac{a_{n-1}}{a_n}\Big) - \mu_n^2.
\end{aligned}
$$

By the expansion (3.9) and (1.7), we then obtain (1.8) by straightforward calculations. A longer expansion gives

$$
\mathbb{V}(X_n) = \sigma^2\Big(n+\frac{3}{2}\Big) - \frac{1}{4} + \frac{\lambda_2}{2\lambda_1^2} + \frac{c_\lambda}{n} + O\big(n^{-2}\big),
$$

where

$$
c_\lambda = \frac{\pi^2}{48} - \frac{1}{4\pi^2} + \frac{19}{48} + \frac{\pi^2}{144\lambda_1^2} - \frac{3\lambda_2}{4\lambda_1^2} + \frac{\pi^2+12}{6}\Big(\frac{\lambda_3}{\lambda_1^3} - \frac{\lambda_2^2}{\lambda_1^4}\Big).
$$

4. Proof of the Stoimenow conjecture (Theorem 4)

We begin by reviewing some necessary definitions on the chord diagrams. A *linearized chord diagram (LCD)*, also known as a Stoimenow matching, is a matching of the set $[2n] = \{1, 2, \ldots, 2n\}$, namely, it is a partition of $[2n]$ into subsets of size exactly two. Each of the subsets is called an *arc*. A matching is a *regular LCD* if it has *no* nested pairs of arcs such that either the openers or the closers are next to each other.

The number of regular linearized chord diagram of size n (length $2n$) equals the n-th Fishburn number f_n; see for instance [6, 30] for the connections and Figure 4.1 for an illustration. By

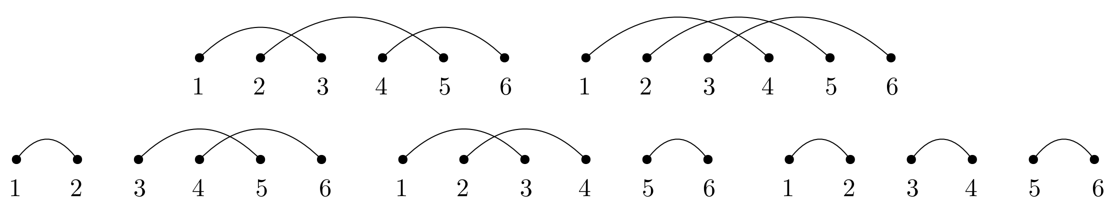


Figure 4.1. *All regular LCDs of size* 3 *where the two on top are connected ones.*

exploiting the relation between the generating function (1.1) of the Fishburn numbers and the "half derivative" of the Dedekind eta-function, Zagier [30] derived the asymptotic behavior of Fishburn numbers f_n, which is our (1.3) with $\lambda_n = 1$ for all n:

$$
f_n := [z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big(1-(1-z)^j\big) = cn^{\frac{1}{2}}\mu^n n!\big(1+O\big(n^{-1}\big)\big), \tag{4.1}
$$

where $(c,\mu) := \big(\frac{12\sqrt{3}}{\pi^{5/2}}\,e^{\frac{\pi^2}{12}}, \frac{6}{\pi^2}\big)$.

Given any regular LCD D, consider a graph G_D (also known as *intersection graph*) whose vertices are arcs of D and two vertices are connected by an edge if the corresponding arcs cross each other; in this case if G_D is connected, then D is also said to be *connected*; see Figure 4.1 for an illustrative example. The corresponding generating function $g(z)$ of connected regular LCDs counted by size is implicitly given in (1.13).

We now prove Theorem 4, showing that the probability of a uniformly generated large random regular LCD being connected is asymptotic to e^{-1}.

Proof. (Theorem 4) We are going to prove

$$\frac{[z^n]g(z)}{[z^n]\Phi(z,0)} = \frac{g_n}{f_n} = e^{-1}\big(1 + O\big(n^{-1}\big)\big), \tag{4.2}$$

where $[z^n]\Phi(z,0) = f_n$ equals the n-th Fishburn number. In view of (4.1), it remains to evaluate g_n as n tends to infinity.

By using the two relations (1.14) and (2.3), we can rewrite (1.13) as

$$\Phi(z,v) = \frac{v}{(1+v)^2} + \frac{1}{(1+v)^2}\sum_{k\geqslant 0}\frac{1}{(1-z)^{k+1}}\prod_{1\leqslant j\leqslant k}\frac{((1-z)^{-j}-1)^2}{1+v(1-z)^{-j}}. \tag{4.3}$$

As a result, the equation $\Phi(z,g(z)) = 1$ can be written as

$$g(z)^2 + g(z) + 1 = W(z) := \sum_{k\geqslant 0}\frac{1}{(1-z)^{k+1}}\prod_{1\leqslant j\leqslant k}\frac{((1-z)^{-j}-1)^2}{1+g(z)(1-z)^{-j}}. \tag{4.4}$$

Taking the coefficients of z^n on both sides yields (for $n \geqslant 1$)

$$[z^n]g(z) + [z^n]g(z)^2 = [z^n]W(z). \tag{4.5}$$

While this equation is still recursive, the dependence of the first-order asymptotic approximation on g is weak in the sense that the dominant asymptotics of the RHS depends only on the coefficient $g'(0) = 1$ (see (1.12)), similar to the two examples in [19, Section 6.1.4].

Technically, comparing the right-hand side of (4.4) with the expression (by (1.1) and (2.3) when $v = 1$)

$$f_n = [z^n]\sum_{k\geqslant 0}\frac{1}{(1-z)^{k+1}}\prod_{1\leqslant j\leqslant k}((1-z)^{-j}-1)^2,$$

we expect that the limiting constant e^{-1} will come from the extra product

$$\overline{W}_k(z) := \prod_{1\leqslant j\leqslant k}\frac{1}{1+g(z)(1-z)^{-j}}.$$

To justify this, we first truncate the series to a polynomial:

$$[z^n]\overline{W}_k(z) = [z^n]\prod_{1\leqslant j\leqslant k}\frac{1}{1+\bar{g}_n(z)(1-z)^{-j}}, \tag{4.6}$$

where $\bar{g}_n(z) := \sum_{1\leqslant \ell\leqslant n} g_\ell z^\ell$. Then we are going to prove that $\bar{g}_n(z) = O(n^{-1})$ when $n|z| \leqslant \mu^{-1}e - \varepsilon$. By the trivial bound $g_n \leqslant f_n$ and the estimate (4.1), we have, when $|z| = \varrho n^{-1}$, $\varrho > 0$,

$$\begin{aligned}|\bar{g}_n(z)| \leqslant \bar{g}_n(|z|) \leqslant \sum_{1\leqslant \ell\leqslant n} f_\ell |z|^\ell &= O\Big(\sum_{1\leqslant \ell\leqslant n}\sqrt{\ell}\,\mu^\ell \ell!\varrho^\ell n^{-\ell}\Big)\\ &= O\Big(\sum_{1\leqslant \ell\leqslant n}\ell\big(\varrho\mu e^{-1}\ell n^{-1}\big)^\ell\Big) = O\big(n^{-1}\big),\end{aligned}$$

whenever $\varrho < \mu^{-1}e$. It follows that, with the same $|z| = \varrho n^{-1}$,

$$\overline{W}_k(z) = O\Big(\exp\Big(|z|\sum_{1\leqslant j\leqslant k}(1-|z|)^{-j}\Big)\Big) = O(1),$$

as long as $\varrho < \mu^{-1}e$. Thus the extra product $\overline{W}_k(z)$ will only affect the constant term in the asymptotic analysis of g_n.

Indeed, from the proof of Theorem 1 or Proposition 6, it suffices to examine the behavior of the finite product in (4.6) when $k = qn + O\big(n^{\frac12+\varepsilon}\big)$ and $z = (\mu n)^{-1}\big(1 + O(n^{-\frac12})\big)$. Note that $q = \mu \log 2 < \mu^{-1}e$. Since $\bar{g}_n(z) = z + O(|z|^2)$ when $z \asymp n^{-1}$, we then get, for such k and z,

$$\begin{aligned}\prod_{1\leqslant j\leqslant k} \frac{1}{1+\bar{g}_n(z)(1-z)^{-j}} &= \exp\bigg(-\bar{g}_n(z)\sum_{1\leqslant j\leqslant k}(1-z)^{-j} + O\bigg(|z|^2\sum_{1\leqslant j\leqslant k}|1-z|^{-2j}\bigg)\bigg)\\ &= \exp\bigg(-\big(z+O(|z|^2)\big)\frac{(1-z)^{-k}-1}{z} + O\big(|z|\big)\bigg)\\ &= e^{-(e^{q/\mu}-1)}\big(1+O\big(n^{-\frac12}\big)\big) = e^{-1}\big(1+O\big(n^{-\frac12}\big)\big).\end{aligned}$$

The detailed proof follows the same procedure that we used for the proof of Theorem 1, and is omitted here. We thus obtain

$$w_n := [z^n]W(z) = c' n^{\frac12}\mu^n n!\big(1+O\big(n^{-1}\big)\big), \quad \text{with} \quad c' := \tfrac{12\sqrt{3}}{\pi^{5/2}}\, e^{\frac{\pi^2}{12}-1}. \tag{4.7}$$

This gives an approximation of the right-hand-side of (4.5). It remains to prove that $g_n = [z^n]g(z)$ is asymptotically equivalent to w_n. Observe first that the unique solution of (4.4) satisfying $g(0) = 0$ and $W(0) = 1$ equals

$$g(z) = -\frac12 + \frac12(4W(z)-3)^{\frac12} = \sum_{\ell\geqslant 1}\frac{(-1)^{\ell-1}}{\ell}\binom{2\ell-2}{\ell-1}(W(z)-1)^{\ell}.$$

Taking the n-th coefficient on both sides yields

$$\begin{aligned}[z^n]g(z) &= \sum_{1\leqslant \ell\leqslant n}\frac{(-1)^{\ell-1}}{\ell}\binom{2\ell-2}{\ell-1}[z^n](W(z)-1)^{\ell}\\ &= w_n - \sum_{1\leqslant j<n} w_j w_{n-j} + 2\sum_{\substack{j_1+j_2+j_3=n\\ 1\leqslant j_1,j_2,j_3<n}} w_{j_1}w_{j_2}w_{j_3} - \cdots .\end{aligned}\tag{4.8}$$

Here the central binomial coefficients in (4.8) only increase exponentially, while w_n grows factorially. Now, by (4.7),

$$\begin{aligned}\sum_{1\leqslant j<n} w_j w_{n-j} &= 2w_{n-1} + O\bigg(\sqrt{n}\,\mu^n\sum_{2\leqslant j\leqslant n-2} j!(n-j)!\bigg)\\ &= O\big(n^{-1}w_n + \sqrt{n}\,\mu^n(n-1)!\big) = O\big(n^{-1}w_n\big),\end{aligned}$$

because $j!(n-j)!$ decreases in $j \in [0, \frac12 n]$. Similarly, $[z^n](W(z)-1)^{\ell} = O\big(n^{-\ell+1}w_n\big)$ for other $\ell \geqslant 3$, showing that (4.8) is itself an asymptotic expansion. Thus $g_n = w_n(1 + O(n^{-1}))$, which, together with (4.7) and (4.1), proves the limiting ratio (4.2), and thus Theorem 4. $\square$

Finer approximations for the ratio $\frac{g_n}{f_n}$ can also be derived by the same saddle-point analysis; for example, we have

$$\frac{g_n}{f_n} = e^{-1}\Big(1 - \frac{\pi^2}{8n} + O\big(n^{-2}\big)\Big).$$

5. Size distribution of random FMs

We prove in this section Theorem 3, which is an extension of the open problem 5.5 by Jelínek [20]. Unlike the analytic proof for Theorem 1, our approach to Theorem 3 builds on a simple partial fraction decomposition. We also briefly discuss a few other sequences of a similar nature.

5.1. **Asymptotic normality of the size.** We recall that the generating function $F(z, v)$ of Λ-FMs is given by (2.3). Our study of the size distribution is restricted to the situation when $\Lambda(z)$ is a polynomial. In the special case when $\Lambda(z) = 1 + z$, the size of an m-dimensional primitive FM lies between m (when only the entries on the main diagonal are 1) and $\binom{m+1}{2}$ (when all entries are 1). Our limit result says roughly that only near the median size $\frac{1}{4}m(m + 3)$ does the number of primitive FMs of dimension m reach its peak among all other possible sizes, which is also the case when 0's and 1's are allowed to appear equally likely in each entry (except the diagonal). For instance, in Figure 1.2, most primitive FMs of dimension 3 have size 4 or 5.

Proof. (Theorem 3) Let Y_m be the size of a random $m \times m$ Λ-FM when all Λ-FMs of dimension m are equally likely to be selected. The corresponding probability generating function of Y_m is given by

$$\mathbb{E}(z^{Y_m}) = \frac{[v^m]F(z, v)}{[v^m]F(1, v)},$$

with F given in (1.4). By partial fraction expansion, we have

$$\mathcal{G}(v) := \prod_{1\leqslant j\leqslant k} \frac{\Lambda(z)^j - 1}{1 + v(\Lambda(z)^j - 1)} = \sum_{1\leqslant j\leqslant k} \frac{C_{k,j}(z)}{1 + v(\Lambda(z)^j - 1)}, \tag{5.1}$$

where $C_{k,j}(z)$ is the residue of $\mathcal{G}(z)$ at $v = (1 - \Lambda(z)^j)^{-1}$ and is given by

$$\begin{aligned}
C_{k,j}(z) &= \big(\Lambda(z)^j - 1\big)^k \prod_{l\neq j, 1\leqslant l\leqslant k} \frac{\Lambda(z)^l - 1}{\Lambda(z)^j - \Lambda(z)^l},\\
&= \big(\Lambda(z)^j - 1\big)^k \Bigg(\prod_{1\leqslant l<j} \frac{\Lambda(z)^l - 1}{\Lambda(z)^j - \Lambda(z)^l}\Bigg)\Bigg(\prod_{j<l\leqslant k} \frac{\Lambda(z)^l - 1}{\Lambda(z)^j - \Lambda(z)^l}\Bigg)\\
&= (-1)^{k-j}\big(\Lambda(z)^j - 1\big)^k \Lambda(z)^{-\binom{j}{2}} \prod_{1\leqslant l\leqslant k-j} \frac{1 - \Lambda(z)^{-j-l}}{1 - \Lambda(z)^{-l}}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
[v^m]F(z, v) &= \sum_{1\leqslant k\leqslant m} \sum_{1\leqslant j\leqslant k} C_{k,j}(z)(-1)^{m-k}\big(\Lambda(z)^j - 1\big)^{m-k}\\
&= \sum_{1\leqslant k\leqslant m} \sum_{1\leqslant j\leqslant k} (-1)^{m-j}\big(\Lambda(z)^j - 1\big)^m \Lambda(z)^{-\binom{j}{2}} \Bigg(\prod_{1\leqslant l\leqslant k-j} \frac{1 - \Lambda(z)^{-l-j}}{1 - \Lambda(z)^{-l}}\Bigg).
\end{aligned}$$

Interchanging the two sums and rearranging the sum-indices lead to

$$[v^m]F(z, v) = \Lambda(z)^{\binom{m+1}{2}} \sum_{0\leqslant j<m} (-1)^j\big(1 - \Lambda(z)^{-m+j}\big)^m \Lambda(z)^{-\binom{j+1}{2}} \sum_{0\leqslant k\leqslant j} \prod_{1\leqslant l\leqslant k} \frac{1 - \Lambda(z)^{-l-m+j}}{1 - \Lambda(z)^{-l}}.$$

For our limit law purposes, we consider $z \sim 1$. Since $\Lambda(z)$ is a polynomial with positive coefficients and $\Lambda(1) > 1$, there is a small neighborhood of unity, say $|z-1| \leqslant \delta$, $\delta > 0$, where $|\Lambda(z)| > 1$. Thus for such z

$$\big(1-\Lambda(z)^{-m+j}\big)^m - 1 = O\big(m|\Lambda(z)|^{-m+j}\big).$$

We then deduce that

$$[v^m]F(z,v) = H(z)\Lambda(z)^{\binom{m+1}{2}}\big(1+O\big(m|\Lambda(z)|^{-m}\big)\big), \tag{5.2}$$

uniformly for $|z-1| \leqslant \delta$, where

$$H(z) := \sum_{j\geqslant 0}(-1)^j\Lambda(z)^{-\binom{j+1}{2}} \sum_{0\leqslant k\leqslant j}\prod_{1\leqslant l\leqslant k}\frac{1}{1-\Lambda(z)^{-l}}.$$

In particular, the total number of m-dimensional Λ-FMs satisfies

$$\frac{[v^m]F(1,v)}{\Lambda(1)^{\binom{m+1}{2}}} \to H(1) = \sum_{j\geqslant 0}(-1)^j\Lambda(1)^{-\binom{j+1}{2}}\sum_{0\leqslant k\leqslant j}\frac{1}{Q_k}, \tag{5.3}$$

where $Q_k := \prod_{1\leqslant \ell\leqslant k}\frac{1}{1-\Lambda(1)^{-\ell}}$. An alternative expression of $H(1)$ with additional numerical advantages is

$$H(1) = \frac{1}{Q_\infty}\sum_{j\geqslant 0}(-1)^j\Lambda(1)^{-\binom{j+1}{2}}\sum_{l\geqslant 0}\frac{(-1)^j\Lambda(1)^{-\binom{l}{2}}}{Q_l}\cdot\frac{1-\Lambda(1)^{-l(j+1)}}{1-\Lambda(1)^{-l}},$$

which follows from the Euler identity

$$\frac{Q_\infty}{Q_k} = \sum_{l\geqslant 0}\frac{(-1)^l\Lambda(1)^{-\binom{l}{2}}}{Q_l}\Lambda(1)^{-kl}.$$

Furthermore, it also follows from (5.2) that

$$\mathbb{E}(z^{Y_m}) = \frac{[v^m]F(z,v)}{[v^m]F(1,v)} = \frac{H(z)}{H(1)}\left(\frac{\Lambda(z)}{\Lambda(1)}\right)^{\binom{m+1}{2}}\big(1+O\big(m\Lambda(1)^{-m}+m|\Lambda(z)|^{-m}\big)\big), \tag{5.4}$$

uniformly for $|z-1| \leqslant \delta$. By applying the Quasi-powers Theorem ([15, IX.5] or [18]), we conclude that the distribution of the random variable Y_m is asymptotically normally distributed with mean and variance asymptotic to

$$\mathbb{E}(Y_m) = \hat{\mu}m(m+1) + \frac{H'(1)}{H(1)} + O\big(m\Lambda(1)^{-m}\big),$$
$$\mathbb{V}(Y_m) = \hat{\sigma}^2 m(m+1) + \frac{H'(1)+H''(1)}{H(1)} - \Big(\frac{H'(1)}{H(1)}\Big)^2 + O\big(m\Lambda(1)^{-m}\big),$$

where $(\hat{\mu}, \hat{\sigma}^2)$ are defined in (1.10). Note that an optimal convergence rate of order $O(m^{-1})$ in the central limit theorem (1.9) is also implied by the same Quasi-Powers Theorem. □

As a special case, consider $\Lambda = \{0, 1, \ldots, h-1\}$, where $h \geqslant 2$. Then $\Lambda(1) = h$. The asymptotic expression (5.3) then suggests the following algorithm for generating a random m-dimensional Λ-FM:

- Generate two independent integer-valued uniform distribution $\mathsf{Uniform}[1, h-1]$ for the two corners on the diagonal, and generate $\mathsf{Uniform}[0, h-1]$ for each of the remaining upper-triangular entries.

- Reject the matrix if it fails to be Fishburn (and then restart the procedure from the previous step), and stop the procedure if it is.

The limiting probability of success is given by, according to (5.3),

$$p_h := \frac{h^2}{(h-1)^2} \sum_{j\geqslant 0} (-1)^j h^{-\binom{j+1}{2}} \sum_{0\leqslant k\leqslant j} \prod_{1\leqslant l\leqslant k} \frac{1}{1-h^{-l}}.$$

While it is expected that the probability p_h tends to 1 as h increases because 0 occurs with less and less probability, the more than doubled jump of the success probability from p_2 to p_3 (see the following table) seems less expected.

h	2	3	4	5	6	7	8	9	10
p_h	0.334	0.706	0.843	0.903	0.935	0.953	0.965	0.972	0.978

These values also show that the naive rejection method is generally very efficient.

Three sequences with different $\Lambda(z)$ are found in the OEIS of the form $[v^m]F(1,v)$, and they are summarized in the following table; their asymptotic behaviors are described by (5.3).

OEIS	[25, A005321]	[25, A289314]	[25, A289315]
$\Lambda(z)$	$1+z$	$1+z+z^2$	$1+z+z^2+z^3$
Generating function $F(1,z)$	$\sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \frac{(2^j-1)z}{1+(2^j-1)z}$	$\sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \frac{(3^j-1)z}{1+(3^j-1)z}$	$\sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \frac{(4^j-1)z}{1+(4^j-1)z}$

5.2. **Andresen and Kjeldsen's 1976 paper.** In a somewhat disguised context of transitively directed graphs, Andresen and Kjeldsen studied in their pioneering paper [1] three sequences connected to primitive FMs, denoted by $\xi_{m,k}$, $\eta_{m,k}$ and $\psi_{m,k}$, respectively. (We change their notation $f(m,k)$ to $f_{m,k}$ to reduce the occurrences of parentheses.) We show in this subsection that these three sequences are all asymptotically normally distributed for large m with mean asymptotic to $\frac{1}{2}m$ and variance to $\frac{1}{4}m$, respectively.

In terms of the matrix language, $\xi_{m,k}$ counts the number of primitive FMs of dimension m with first row sum k, and $\psi_{m,k}$ the number of upper triangular primitive matrices (matrices with entries 0 or 1) of dimension m with first row sum k such that

- the j-th column ($1 \leqslant j < m$) is a zero column if and only if the $(j+1)$-st row is a zero row;
- all nonzero columns and rows form a primitive FM.

Let $P_m^{[f]}(v) = \sum_{k=1}^m f_{m,k} v^k$ with $f = \xi, \eta, \psi$ be the generating polynomial of $f_{m,k}$. While a combinatorial interpretation of the last sequence $\eta_{m,k}$ is still lacking (an open question in [1]), its generating polynomial satisfies a similar type of recurrence as that of ξ and ψ:

$$\begin{cases} P_m^{[\xi]}(v) = vP_{m-1}^{[\xi]}(1+2v) - vP_{m-1}^{[\xi]}(v), \\ P_m^{[\eta]}(v) = vP_{m-1}^{[\eta]}(1+2v) - (1+v)P_{m-1}^{[\eta]}(v), \\ P_m^{[\psi]}(v) = vP_{m-1}^{[\psi]}(1+2v) + (1-v)P_{m-1}^{[\psi]}(v), \end{cases}$$

for $n \geqslant 2$, all with the same initial conditions $P_1^{[\cdot]}(v) = v$. Other types of recurrences are also derived in [1]. These recurrences are readily solved by iterating the corresponding functional equations satisfied by the bivariate generating functions, and we obtain

Sequence	$\xi_{m,k}$	$\eta_{m,k}$	$\psi_{m,k}$
OEIS	[25, A259971]	[25, A259972]	[25, A259970]
Bivariate GF	$\sum_{k\geqslant 0}\prod_{0\leqslant j\leqslant k}\frac{(2^j-1+2^j v)z}{1-z+2^j(1+v)z}$	$\sum_{k\geqslant 0}\prod_{0\leqslant j\leqslant k}\frac{(2^j-1+2^j v)z}{1+2^j(1+v)z}$	$\sum_{k\geqslant 0}\prod_{0\leqslant j\leqslant k}\frac{(2^j-1+2^j v)z}{1-2z+2^j(1+v)z}$
$\sum_k f_{m,k}$	[25, A005321]	[25, A005014]	[25, A005016]

In particular, by a direct partial fraction expansion of the bivariate generating function (similar to the one we carried out for $\mathcal{G}(v)$ in (5.1)), we obtain

$$P_m^{[\eta]}(v) = v\sum_{0\leqslant k<m}(-1)^k(1+v)^{m-1-k}\prod_{k<l<m}(2^l-1) \qquad (m\geqslant 1),$$

and the reason of introducing $\eta_{m,k}$ is because of the relations (see [1, Eq. (9)])

$$P_m^{[\xi]}(v) = \sum_{0\leqslant j<m}\binom{m-1}{j}P_{m-j}^{[\eta]}(v), \quad \text{and} \quad P_n^{[\psi]}(v) = \sum_{0\leqslant j<m}\binom{m-1}{j}2^j P_{m-j}^{[\eta]}(v).$$

From these forms, the limiting normal distribution in all cases can be derived by a similar argument used above for the size distribution Y_m. For example, consider $\eta_{m,k}$ and write $Q_m := \prod_{1\leqslant j\leqslant m}(1-2^{-j})$. Then

$$\begin{aligned} P_m^{[\eta]}(v) &= v2^{\binom{m}{2}}\sum_{0\leqslant k<m}\frac{(-1)^k 2^{-\binom{k+1}{2}}Q_{m-1}}{Q_k}(1+v)^{m-1-k}, \\ &= T(v)v(1+v)^{m-1}2^{\binom{m}{2}}\Big(1+O\Big(\sum_{k\geqslant m}2^{-\binom{k+1}{2}}|1+v|^{-k}\Big)\Big), \end{aligned}$$

where

$$T(v) := Q_\infty\sum_{k\geqslant 0}\frac{(-1)^k 2^{-\binom{k}{2}}}{Q_k}(1+v)^{-k} \quad \text{with} \quad Q_\infty := \prod_{j\geqslant 1}(1-2^{-j}),$$

is a meromorphic function of v. This implies that

$$\frac{P_m^{[\eta]}(v)}{P_m^{[\eta]}(1)} = \frac{vT(v)}{T(1)}\Big(\frac{1+v}{2}\Big)^{m-1}\big(1+O(|1+v|^{-m})\big),$$

uniformly for $|v+1|\geqslant 1+\varepsilon$, and from this we then deduce the asymptotic normality $\mathscr{N}\big(\frac{1}{2}m, \frac{1}{4}m\big)$ for the underlying random variables by the Quasi-powers Theorem ([15, IX.5] or [18]). Exactly the same type of results holds for the other two sequences.

5.3. **Some related OEIS sequences.** A few other sequences in the OEIS are closely connected to the sequences we discussed in this section. We list them in the following table. Asymptotic or distributional properties can be dealt with by the same techniques, and are omitted here.

[25, A005327]	$\frac{1}{1+z}\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\frac{(2^j-1)z}{1+2^jz}$	[25, A002820]	$2^{\binom{n}{2}}\times \text{A005327}(n+1)$
[25, A005016]	$\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\frac{(2^j-1)z}{1+(2^j-2)z}$	[25, A005331]	$\frac{1}{1-z}\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\frac{(2^j-1)z}{1+(2^j-2)z}$
[25, A005329]	$\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\frac{2^jz}{1+2^jz}$	[25, A028362]	$\sum_{k\geqslant 0}\prod_{0\leqslant j<k}\frac{2^jz}{1-2^jz}$
[25, A182507]	$\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\frac{j2^{j-1}z}{1+j2^jz}$	[25, A006116]:	$\sum_{k\geqslant 0}z^k\prod_{0\leqslant j\leqslant k}\frac{1}{1-2^jz}$

6. Self-dual FMs and FMs without 1's

In this section, we briefly describe the limiting behaviors of random self-dual FMs and random FMs whose smallest nonzero entries are 2, respectively. The asymptotics in both cases are similar to FMs that we discussed above but involve a stretched exponential factor of the form $e^{\Theta(\sqrt{n})}$. We only sketch the proof in the self-dual case, and omit that in the other.

6.1. **Dimension of self-dual FMs.** The dimension distribution of Λ-FMs (Theorem 1) exhibits a limiting invariance property in the sense that the central limit theorem is independent of the entry-set Λ as long as $\lambda_1 > 0$. We show here that the same limiting property holds even when we restrict our random matrices to be self-dual (or persymmetric). What is less expected here is that the variance in the random self-dual FMs is asymptotically double that in the ordinary case (while the mean remains asymptotically the same); see Table 3 for a numerical illustration in the case of primitive FMs (with $\Lambda(z) = 1 + z$).

Self-dual primitive FMs

$n\backslash k$	1	2	3	4	5	6	(μ_n, σ_n^2)
1	1						$(1, 0)$
2		1					$(2, 0)$
3		1	1				$(\frac{5}{2}, \frac{1}{4})$
4			2	1			$(\frac{10}{3}, \frac{2}{9})$
5			2	3	1		$(\frac{23}{6}, \frac{17}{36})$
6			1	5	6	1	$(\frac{59}{13}, \frac{94}{169})$

Primitive FMs

$n\backslash k$	1	2	3	4	5	6	(μ_n, σ_n^2)
1	1						$(1, 0)$
2		1					$(2, 0)$
3		1	1				$(\frac{5}{2}, \frac{1}{4})$
4			4	1			$(\frac{16}{5}, \frac{4}{25})$
5			4	11	1		$(\frac{61}{16}, \frac{71}{256})$
6			1	33	26	1	$(\frac{271}{61}, \frac{1162}{3721})$

Table 3. *The first few values of the dimension statistics in self-dual primitive FMs and ordinary primitive FMs (where n denotes the size and k the dimension). In particular, among the 5 primitive FMs of size 4, only 3 are self-dual, resulting in higher variance, and a similar observation holds for matrices of larger size.*

Theorem 18. *Let Z_n denote the dimension of a random self-dual Λ-FM, where all size-n self-dual Λ-FMs are equally likely. If $\Lambda(z)$ is analytic at $z = 0$ with $\lambda_1 > 0$, then Z_n is asymptotically normal:*

$$\frac{Z_n - \mu n - \mu'\sqrt{n}}{\sigma\sqrt{2n}} \xrightarrow{d} \mathscr{N}(0, 1),$$

where (μ, σ) is defined in (1.5), $\mu' := \frac{\sqrt{6}}{2\pi^3}\big(12\log 2 - \pi^2\sqrt{\lambda_1}\big)$, and the mean and the variance are asymptotic to

$$
\begin{aligned}
&\mathbb{E}(Z_n) = \mu n + \mu'\sqrt{n} + O(1),\\
(6.1)\qquad &\mathbb{V}(Z_n) = 2\sigma^2 n + \frac{\sqrt{6}}{4\pi^5}\big(24(18-\pi^2)\log 2 - \pi^2(24-\pi^2)\sqrt{\lambda_1}\big)\sqrt{n} + O(1),
\end{aligned}
$$

respectively.

Proof. Our analysis is based on the generating function $G(z,v)$ of the dimension (marked by v) of self-dual FMs of a given size (marked by z):

$$
(6.2)\qquad G(z,v) + v = \sum_{k\geqslant 1} \Lambda(z)^k \Lambda(z^2)^{\binom{k}{2}} v^{2k-1} \frac{1+v(\Lambda(z^2)^k - 1)}{\Lambda(z^2)^k - 1} \prod_{1\leqslant j\leqslant k} \frac{\Lambda(z^2)^j - 1}{1+v^2(\Lambda(z^2)^j - 1)}.
$$

This is obtained by substituting $1+z$ by $\Lambda(z)$ in the generating function of primitive self-dual FMs derived by Jelínek [20]. When $v = 1$, we have

$$
G(z,1) + 1 = \sum_{k\geqslant 1} G_k(z), \quad \text{with} \quad G_k(z) := \Lambda(z)^k \prod_{1\leqslant j<k} \big(\Lambda(z^2)^j - 1\big).
$$

Asymptotic approximation of $[z^n]G(z,1)$ was already derived in [19]; in particular, when $\lambda_1 > 0$,

$$
[z^n]G(z,1) = c e^{\beta\sqrt{n}} (\lambda_1 \mu e^{-1})^{\frac{1}{2}n} n^{\frac{1}{2}(n+1)} \big(1 + O\big(n^{-\frac{1}{2}}\big)\big),
$$

where

$$
(c, \beta, \mu) := \Big(\tfrac{3\sqrt{2}}{\pi^{3/2}} 2^{\frac{\lambda_2}{\lambda_1} - \frac{\lambda_1}{2}} e^{-\frac{\lambda_1}{4} - \frac{\pi^2}{24} + \frac{\pi^2\lambda_2}{12\lambda_1^2} + \frac{3\lambda_1}{2\pi^2}(\log 2)^2}, \tfrac{\sqrt{6\lambda_1}}{\pi}\log 2, \tfrac{6}{\pi^2}\Big).
$$

A finer expansion can be derived, which is of the form

$$
(6.3)\qquad [z^n]\sum_{k\geqslant 1} \Lambda(z)^k \prod_{1\leqslant j<k} \big(\Lambda(z^2)^j - 1\big) = c e^{\beta\sqrt{n}} (\lambda_1 \mu e^{-1})^{\frac{1}{2}n} n^{\frac{1}{2}(n+1)} \bigg(1 + \sum_{j\geqslant 1} \bar{d}_j n^{-\frac{1}{2}j}\bigg),
$$

for some computable coefficients $\bar{d}_j$.

We now compute the mean $\mathbb{E}(Z_n)$. For convenience, write $\Lambda_j := \Lambda(z^j)$. Taking the derivative with respect to v and substituting $v = 1$ on both sides of (6.2) give

$$
\begin{aligned}
M_1(z) := \partial_v G(z,v)|_{v=1} &= \sum_{k\geqslant 1} G_k(z) \Big(\frac{2}{\Lambda_2 - 1} - \frac{\Lambda_2^{-k}(\Lambda_2 + 1)}{\Lambda_2 - 1}\Big) - 1\\
&= \frac{2(G(z,1)+1)}{\Lambda_2 - 1} - \frac{\Lambda_2 + 1}{\Lambda_2 - 1} \sum_{k\geqslant 1} \frac{\Lambda_1^k}{\Lambda_2^k} \prod_{1\leqslant j<k} \big(\Lambda_2^j - 1\big) - 1.
\end{aligned}
$$

Let now

$$
S_1(z) := \sum_{k\geqslant 1} \frac{\Lambda_1^k}{\Lambda_2^k} \prod_{1\leqslant j<k} \big(\Lambda_2^j - 1\big) = \sum_{k\geqslant 0} \frac{\Lambda_1^{k+1}}{\Lambda_2^{k+1}} \prod_{1\leqslant j\leqslant k} \big(\Lambda_2^j - 1\big).
$$

Then

$$S_1(z) = \frac{\Lambda_1}{\Lambda_2} + \frac{\Lambda_1}{\Lambda_2}\sum_{k\geqslant 1}\frac{\Lambda_1^k(\Lambda_2^k-1)}{\Lambda_2^k}\prod_{1\leqslant j<k}\left(\Lambda_2^j-1\right)$$
$$= \frac{\Lambda_1}{\Lambda_2}(G(z,1)+2) - \frac{\Lambda_1}{\Lambda_2}\,S_1(z).$$

Thus $S_1(z)$ is solved to be

$$S_1(z) = \frac{\Lambda_1}{\Lambda_1+\Lambda_2}(G(z,1)+2),$$

and, consequently,

$$M_1(z) = \frac{2\Lambda_2-\Lambda_1(\Lambda_2-1)}{(\Lambda_2-1)(\Lambda_1+\Lambda_2)}(G(z,1)+2) - \frac{2}{\Lambda_2-1} - 1.$$

By (6.3) and the argument used in the proof of Theorem 2, we then deduce the asymptotic approximation of the mean (6.1).

For the variance, we begin by computing the generating function corresponding to the second moment:

$$M_2(z) := \partial_v^2 G(z,v)|_{v=1} + \partial_v G(z,v)|_{v=1}$$
$$= \sum_{k\geqslant 1} G_k(z)\left(\frac{4(\Lambda_2^2+1)\Lambda_2^{-2k}}{(\Lambda_2-1)(\Lambda_2^2-1)} + \frac{(\Lambda_2^2-2\Lambda_2-7)\Lambda_2^{-k}}{(\Lambda_2-1)^2} - \frac{4(\Lambda_2^2-2\Lambda_2-1)}{(\Lambda_2-1)(\Lambda_2^2-1)}\right) - 1.$$

Let

$$S_2(z) := \sum_{k\geqslant 1}\frac{\Lambda_1^k}{\Lambda_2^{2k}}\prod_{1\leqslant j<k}\left(\Lambda_2^j-1\right).$$

Then

$$S_2(z) = \frac{\Lambda_1}{\Lambda_2^2} + \frac{\Lambda_1^2(\Lambda_2-1)}{\Lambda_2^4} + \frac{\Lambda_1^2}{\Lambda_2^3}\sum_{k\geqslant 1}\Lambda_1^k\left(1-\frac{\Lambda_2+1}{\Lambda_2^{k+1}}+\frac{1}{\Lambda_2^{2k+1}}\right)\prod_{1\leqslant j<k}\left(\Lambda_2^j-1\right)$$
$$= \frac{\Lambda_1}{\Lambda_2^2} + \frac{\Lambda_1^2(\Lambda_2-1)}{\Lambda_2^4} + \frac{\Lambda_1^2}{\Lambda_2^3}\left(G(z,1)+1-\frac{\Lambda_2+1}{\Lambda_2}\,S_1(z)+\frac{S_2(z)}{\Lambda_2}\right),$$

which is the solved to be

$$S_2(z) = \frac{\Lambda_1^2}{(\Lambda_1+\Lambda_2)(\Lambda_1+\Lambda_2^2)}\,(G(z,1)+2) + \frac{\Lambda_1}{\Lambda_1+\Lambda_2^2}.$$

It follows that

$$M_2(z) = \left(\frac{\Lambda_1}{\Lambda_1+\Lambda_2} + \frac{4\Lambda_2}{(\Lambda_2-1)^2(\Lambda_2^2-1)}\left(\frac{3\Lambda_2^2-1}{\Lambda_1+\Lambda_2} - \frac{\Lambda_2^2(\Lambda_2^2+1)}{\Lambda_1+\Lambda_2^2}\right)\right)(G(z,1)+2)$$
$$+ \frac{8\Lambda_2}{\Lambda_2^2-1} - \frac{4\Lambda_2^2(\Lambda_2^2+1)}{(\Lambda_2-1)(\Lambda_2^2-1)(\Lambda_1+\Lambda_2^2)} - 1.$$

From this expression and the expansion (6.3), we deduce an asymptotic approximation to the second moment $\mathbb{E}(Z_n^2)$, and then the asymptotic variance in (6.1).

The proof for the normal limit law is similar to that of Theorem 1, with the modifications needed to incorporate the change at the order $\sqrt{n}$. We list here the major steps. Prove first that when $\Lambda(z) = e^z$ and $v = e^{\frac{i\vartheta}{\sqrt{n}}}$,

$$[z^n]G(z, v) = ce^{\beta\sqrt{n}}\rho^{\frac{1}{2}n}n^{\frac{1}{2}(n+1)}e^{(\mu\sqrt{n}+\mu')i\vartheta-\sigma^2\vartheta^2}\left(1 + O\left(n^{-\frac{1}{2}}\right)\right),$$

where

$$(c, \beta, \rho) = \left(\tfrac{3\sqrt{2}}{\pi^{3/2}}\, e^{-\frac{1}{4}+\frac{3}{2\pi^2}(\log 2)^2}, \sqrt{\mu}\,\log 2, \tfrac{\mu}{e}\right).$$

Then, by the change of variables $\Lambda(z^2) = e^{y^2}$, and by following the same analysis, we deduce that

$$[z^n]G(z, v) = ce^{\beta\sqrt{n}}\rho^{\frac{1}{2}n}n^{\frac{1}{2}(n+1)}e^{(\mu\sqrt{n}+\mu')i\vartheta-\sigma^2\vartheta^2}\left(1 + O\left((|\vartheta| + |\vartheta|^3)n^{-\frac{1}{2}}\right)\right),$$

uniformly for $|\vartheta| = O\left(n^{-\frac{1}{2}}\right)$, where

$$(c, \beta, \rho) = \left(\tfrac{3\sqrt{2}}{\pi^{3/2}}\, 2^{\frac{\lambda_2}{\lambda_1}-\frac{\lambda_1}{2}}\, e^{-\frac{\lambda_1}{4}-\frac{\pi^2}{24}+\frac{\pi^2\lambda_2}{12\lambda_1^2}+\frac{3\lambda_1}{2\pi^2}(\log 2)^2}, \sqrt{\mu\lambda_1}\,\log 2, \tfrac{\mu\lambda_1}{e}\right).$$

This leads to the asymptotic normality of Z_n in Theorem 18. $\square$

6.2. **FMs without 1's.** What happens if $\lambda_1 = 0$ and the smallest nonzero entry is 2? In this case, all generating functions remain the same but with $\Lambda(z) = 1+\lambda_2 z^2+\cdots$. Following the asymptotic approximations derived in [19] and the techniques used above, we can also prove the corresponding central limit theorem for the dimension of random FMs without 1's.

Theorem 19. *Assume that $\Lambda(z)$ is analytic at $z = 0$ with $\lambda_1 = 0$, $\lambda_2 > 0$ and that all such Λ-FMs of size n are equally likely to be selected. Then the dimension X_n of a random matrix is asymptotically normally distributed with mean and variance both linear in n:*

$$\frac{X_n - \bar{\mu}n - \bar{\mu}'\sqrt{n}}{\bar{\sigma}\sqrt{n}} \xrightarrow{d} \mathscr{N}(0, 1), \quad \text{with} \quad (\bar{\mu}, \bar{\mu}', \bar{\sigma}^2) := \left(\frac{3}{\pi^2}, -\frac{\sqrt{3}\lambda_3}{2\pi\lambda_2^{3/2}}, \frac{3(12-\pi^2)}{2\pi^4}\right), \tag{6.4}$$

so that $\bar{\mu} = \frac{1}{2}\mu$ and $\bar{\sigma}^2 = \frac{1}{2}\sigma^2$, where (μ, σ^2) is given in (1.5), and

$$\mathbb{E}(X_n) = \bar{\mu}n + \bar{\mu}'\sqrt{n} + O(1),$$
$$\mathbb{V}(X_n) = \bar{\sigma}^2 n + \left(\frac{\lambda_3(\pi^2-6)}{4\pi^2\lambda_2^{3/2}} - \frac{2\lambda_5}{\lambda_2^{5/2}} + \frac{4\lambda_3\lambda_4}{\lambda_2^{7/2}} - \frac{2\lambda_3^3}{\lambda_2^{9/2}}\right)\sqrt{\bar{\mu}n} + O(1),$$

respectively.

Both the mean and the variance constants are halved, when compared with random FMs with $\lambda_1 > 0$ (Theorem 1). This is intuitively clear as one expects that the entry 2 is omnipresent.

The analysis of this well anticipated limit result is much more involved than it looks because the polynomial term in the asymptotic approximation depends on the first nonzero odd number in the entry-set Λ. More precisely, if $\lambda_{2j-1} = 0$, for $1 \leqslant j \leqslant \ell$, and $\lambda_2, \lambda_{2\ell+1} > 0$, then it is proved in [19] that the total number of Λ-FMs of size n satisfies

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left(1-\Lambda(z)^{-j}\right) = c_\ell e^{\beta\sqrt{n}}\rho^{\frac{1}{2}n}n^{\frac{1}{2}n+\chi_\ell}\left(1 + O\left(n^{-\frac{1}{2}}\right)\right),$$

where (c_ℓ, χ_ℓ) depends not only on ℓ but also on the parity of n, and $(\beta, \rho) = \left(\frac{\lambda_3\pi}{2\sqrt{3}\lambda_2^{3/2}}, \frac{3\lambda_2}{e\pi^2}\right)$.

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Institute of Statistical Science, Academia Sinica, Taipei, 115, Taiwan

Email address: hkhwang@stat.sinica.edu.tw

School of Mathematical Sciences, Xiamen University, Xiamen, 361005 P.R. China; Institute für Diskrete Mathematik und Geometrie, Techische Universität Wien, Wiedner Hauptstr. 8–10, 1040 Vienna, Austria
Email address: `yjin@xmu.edu.cn; yu.jin@univie.ac.at`

Fakultät für Mathematik, Universität Wien, Vienna, Austria
Email address: `michael.schlosser@univie.ac.at`