LES CAHIERS DE PHILIPPE FLAJOLET

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(with Brigitte Vallée, Julien Clément, . . .)

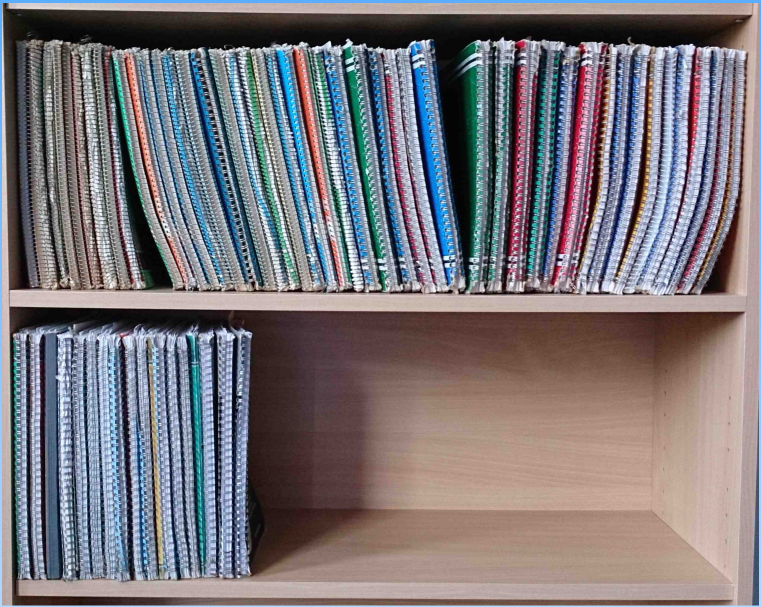
June 19, 2014

Cigarettes

$\Downarrow$

Theory

OUTSIDE LOOK: 65 NOTEBOOKS



VERY STRONG "FLAVORS"

✔ Cigarette smoke

✔ Pen ink

CONTENTS

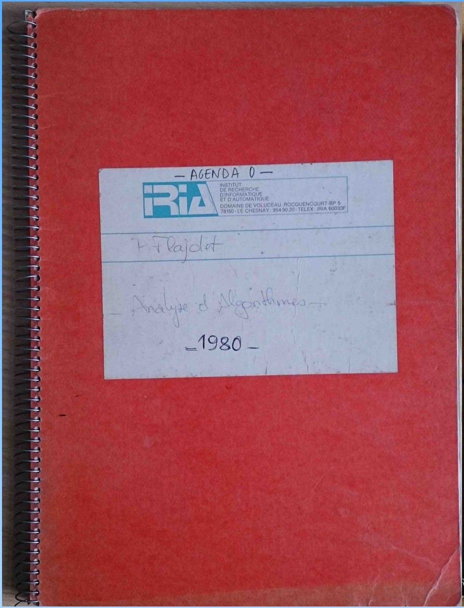
All academic: evolution of ideas, dvpmt of techniques

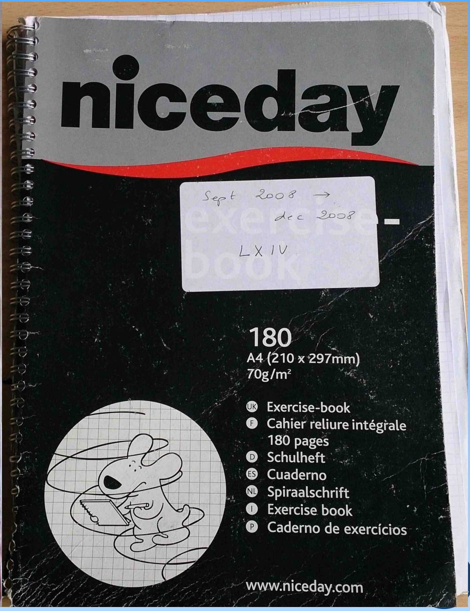
- **Notes/Summaries for talks, lectures, courses, . . .**
- **Drafts for papers, book chapters, preprints, . . .**
- **Work summary**
- **Maple calculations (symbolic, numerical, figures, tables, expansions)**
- **Email/letter correspondences**
- **Miscellaneous**

Three categories

- **Finished, well-explored techniques/topics**
- **Unfinished**
- **We-don’t-know-yet**

AGENDA 0 & CAHIER LXIV





SNAPSHOTS

23 Mars 1980

Les nombres de Stirling de seconde espèce : séries ordinaires et exponentielle

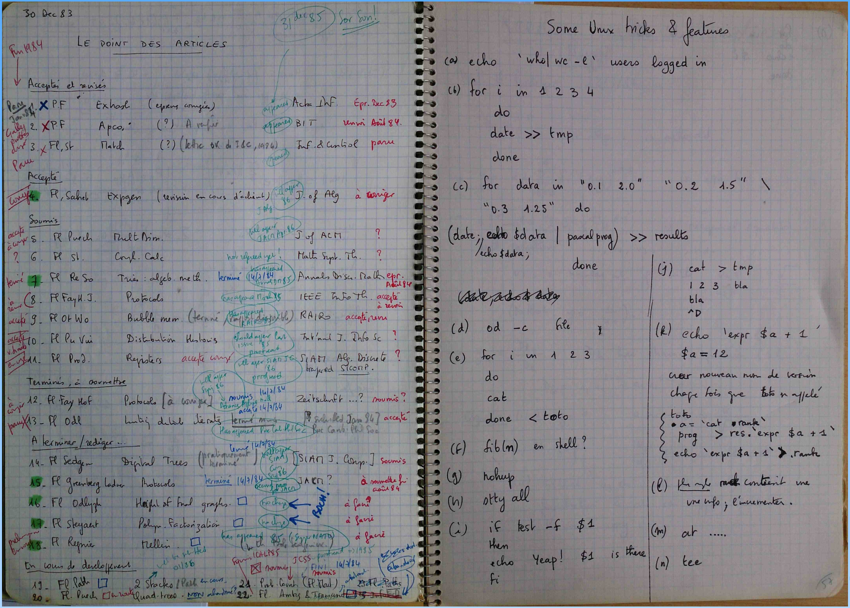
$S_{n,k}$ désigne le nombre de partitions de $[n]$ en k classes (blocs) ;
$k!\, S_{n,k}$ représente ainsi le nombre de surjections de $[n]$ sur $[k]$.
Les deux expressions de séries génératrices (ordinaire ou exponentielle) des $S_{n,k}$ correspondent à deux présentations différentes des partitions d'ensembles.

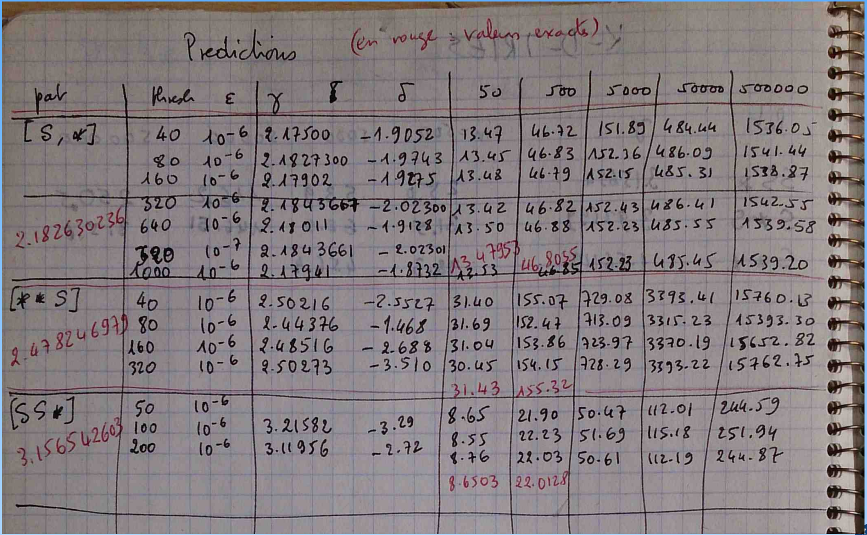
1) $$\sum S_{n,k} \frac{z^n}{n!} = \frac{(e^z-1)^k}{k!}$$

2) $$\sum S_{n,k}\, z^n = \frac{z^k}{(1-z)(1-2z)\cdots(1-kz)}$$

Une preuve directe de ces deux séries génératrices correspond à une preuve combinatoire que 2) est la transformée de Laplace-Borel de 1.
Pour 1), la preuve résulte des manipulations classiques des séries génératrices associées à des mots (cf Analyse d'Algorithmes 1980)

SNAPSHOTS

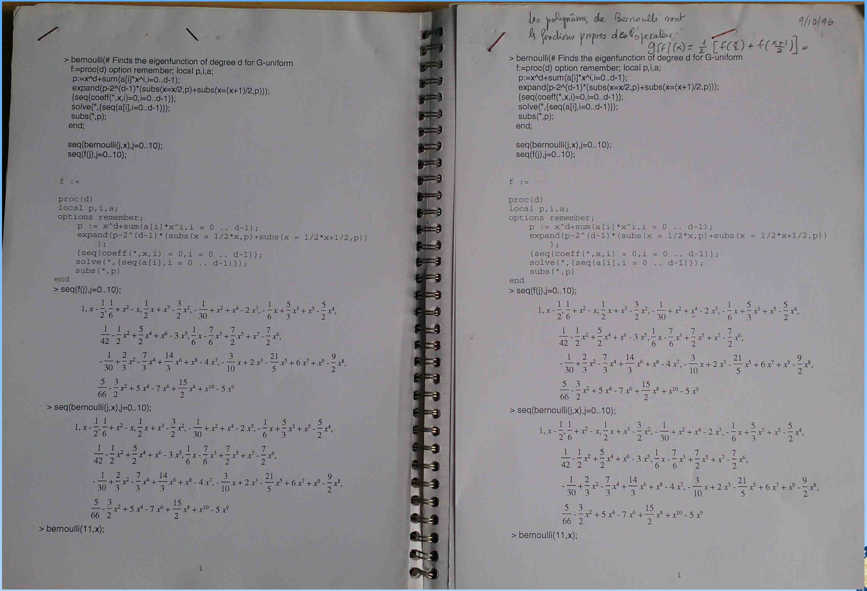

30 Dec 83
LE POINT DES ARTICLES
Acceptés et revisés
Soumis
Terminés, à soumettre
A terminer / rediger ...
En cours de developpement
Some Unix tricks & features
(a) echo `who | wc -l` users logged in
(b) for i in 1 2 3 4
do
date >> tmp
done
(d) od -c file
(e) for i in 1 2 3
do
cat
done < toto
(f) fib(n) en shell ?
(g) nohup
(h) stty all
(i) if test -f $1
then
echo Yeap! $1 is there
fi
(j) cat > tmp
(m) ar .....
(n) tee



Predictions (en rouge : valeurs exactes)

par	thresh	ε	γ	δ	50	500	5000	50000	500000
[S, *]	40	10^{-6}	2.17500	−1.9052	13.47	46.72	151.89	484.44	1536.05
	80	10^{-6}	2.1827300	−1.9743	13.45	46.83	152.36	486.09	1541.44
	160	10^{-6}	2.17902	−1.9275	13.48	46.79	152.15	485.31	1538.87
	320	10^{-6}	2.1843667	−2.02300	13.42	46.82	152.43	486.41	1542.55
2.182630236	640	10^{-6}	2.18011	−1.9128	13.50	46.88	152.23	485.55	1539.58
	720	10^{-7}	2.1843661	−2.02301	13.47953	46.8055			
	1000	10^{-6}	2.17941	−1.8732	13.53	46.85	152.23	485.45	1539.20
[* * S]	40	10^{-6}	2.50216	−2.5527	31.40	155.07	729.08	3393.41	15760.13
2.478246979	80	10^{-6}	2.44376	−1.468	31.69	152.47	713.09	3315.23	15393.30
	160	10^{-6}	2.48516	−2.688	31.04	153.86	723.97	3370.19	15652.82
	320	10^{-6}	2.50273	−3.510	30.45	154.15	728.29	3393.22	15762.75
					31.43	155.32			
[S S *]	50	10^{-6}			8.65	21.90	50.47	112.01	244.59
3.156542603	100	10^{-6}	3.21582	−3.29	8.55	22.23	51.69	115.18	251.94
	200	10^{-6}	3.11956	−2.72	8.76	22.03	50.61	112.19	244.87
					8.6503	22.0128			

SNAPSHOTS



```
> bernoulli(# Finds the eigenfunction of degree d for G-uniform
  f:=proc(d) option remember; local p,i,a;
   p:=x^d+sum(a[i]*x^i,i=0..d-1);
   expand(p-2^(d-1)*(subs(x=x/2,p)+subs(x=(x+1)/2,p)));
   {seq(coeff(",x,i)=0,i=0..d-1)};
   solve(",{seq(a[i],i=0..d-1)});
   subs(",p);
  end;

  seq(bernoulli(j,x),j=0..10);
  seq(f(j),j=0..10);


f :=

proc(d)
local p,i,a;
options remember;
    p := x^d+sum(a[i]*x^i,i = 0 .. d-1);
    expand(p-2^(d-1)*(subs(x = 1/2*x,p)+subs(x = 1/2*x+1/2,p))
        );
    {seq(coeff(",x,i) = 0,i = 0 .. d-1)};
    solve(",{seq(a[i],i = 0 .. d-1)});
    subs(",p)
end
> seq(f(j),j=0..10);
```

$$1,\ x-\frac{1}{2},\ \frac{1}{6}+x^2-x,\ \frac{1}{2}x+x^3-\frac{3}{2}x^2,\ -\frac{1}{30}+x^2+x^4-2x^3,\ -\frac{1}{6}x+\frac{5}{3}x^3+x^5-\frac{5}{2}x^4,$$
$$\frac{1}{42}-\frac{1}{2}x^2+\frac{5}{2}x^4+x^6-3x^5,\ \frac{1}{6}x-\frac{7}{6}x^3+\frac{7}{2}x^5+x^7-\frac{7}{2}x^6,$$
$$-\frac{1}{30}+\frac{2}{3}x^2-\frac{7}{3}x^4+\frac{14}{3}x^6+x^8-4x^7,\ -\frac{3}{10}x+2x^3-\frac{21}{5}x^5+6x^7+x^9-\frac{9}{2}x^8,$$
$$\frac{5}{66}-\frac{3}{2}x^2+5x^4-7x^6+\frac{15}{2}x^8+x^{10}-5x^9$$

```
> seq(bernoulli(j,x),j=0..10);
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```
> bernoulli(11,x);
```

1

les polynômes de Bernoulli sont les fonctions propres de l'opérateur $g[f](x)=\frac{1}{2}\left[f\left(\frac{x}{2}\right)+f\left(\frac{x+1}{2}\right)\right]$.

9/10/96


```
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   {seq(coeff(",x,i)=0,i=0..d-1)};
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  end;

  seq(bernoulli(j,x),j=0..10);
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f :=

proc(d)
local p,i,a;
options remember;
    p := x^d+sum(a[i]*x^i,i = 0 .. d-1);
    expand(p-2^(d-1)*(subs(x = 1/2*x,p)+subs(x = 1/2*x+1/2,p))
        );
    {seq(coeff(",x,i) = 0,i = 0 .. d-1)};
    solve(",{seq(a[i],i = 0 .. d-1)});
    subs(",p)
end
> seq(f(j),j=0..10);
```

$$1,\ x-\frac{1}{2},\ \frac{1}{6}+x^2-x,\ \frac{1}{2}x+x^3-\frac{3}{2}x^2,\ -\frac{1}{30}+x^2+x^4-2x^3,\ -\frac{1}{6}x+\frac{5}{3}x^3+x^5-\frac{5}{2}x^4,$$
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```
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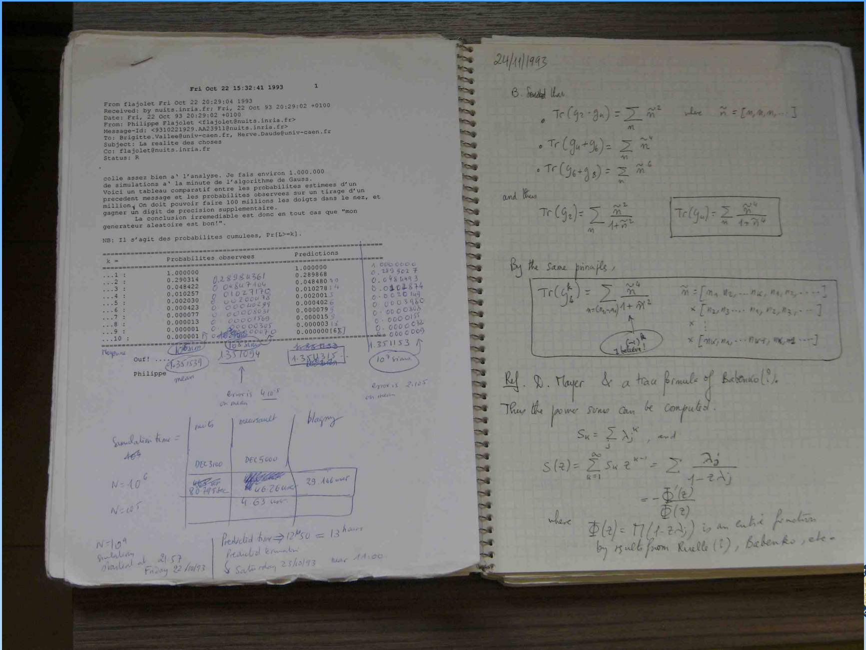
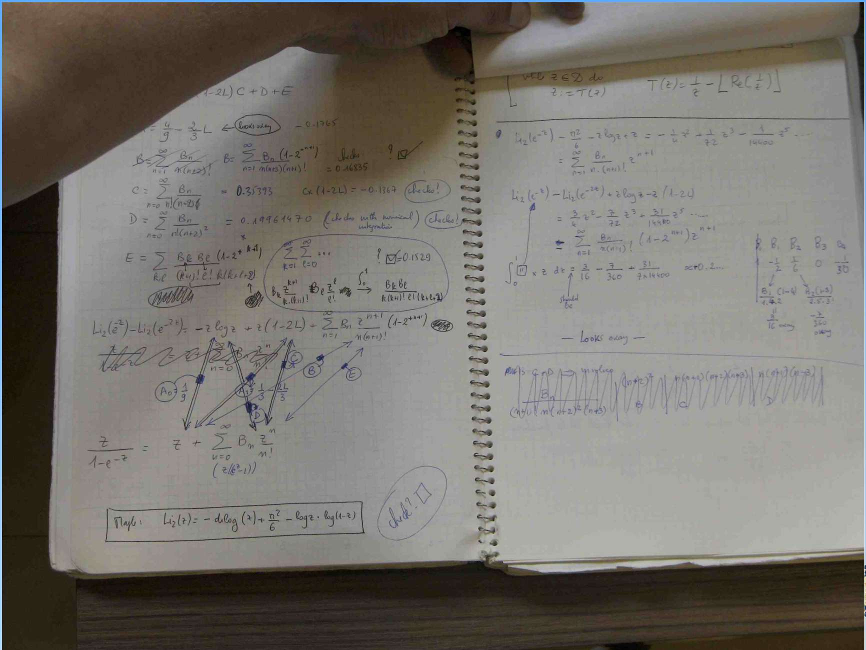
```

1

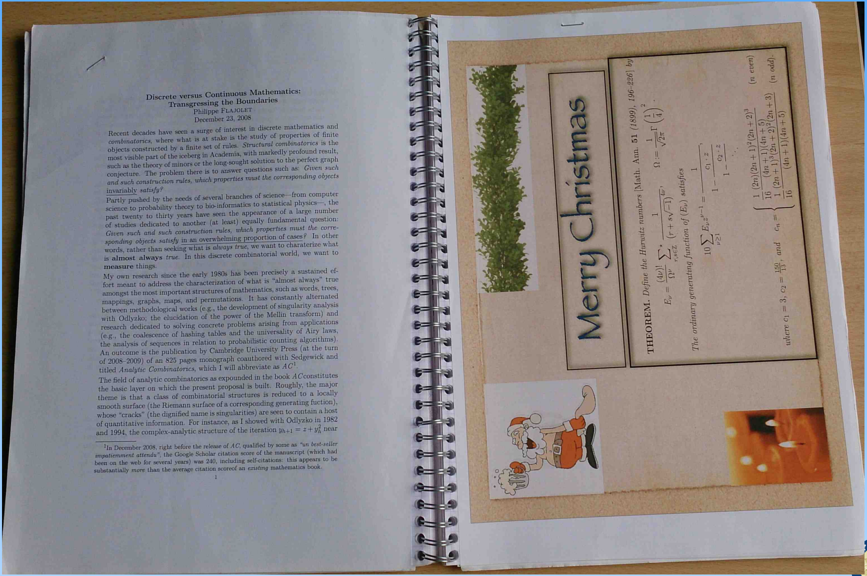
SNAPSHOTS


Fri Oct 22 15:32:41 1993
From flajolet Fri Oct 22 20:29:04 1993
Received: by nuits.inria.fr; Fri, 22 Oct 93 20:29:02 +0100
Date: Fri, 22 Oct 93 20:29:02 +0100
From: Philippe Flajolet <flajolet@nuits.inria.fr>
Message-Id: <9310221929.AA23911@nuits.inria.fr>
To: Brigitte.Vallee@univ-caen.fr, Herve.Daude@univ-caen.fr
Subject: La realite des choses
Cc: flajolet@nuits.inria.fr
Status: R
colle assez bien a' l'analyse. Je fais environ 1.000.000
de simulations a' la minute de l'algorithme de Gauss.
Voici un tableau comparatif entre les probabilites estimees d'un
precedent message et les probabilites observees sur un tirage d'un
million. On doit pouvoir faire 100 millions les doigts dans le nez, et
gagner un digit de precision supplementaire.
La conclusion irremediable est donc en tout cas que "mon
generateur aleatoire est bon!".
NB: Il s'agit des probabilites cumulees, Pr[L>=k].
k = Probabilites observees Predictions
...1 : 1.000000 1.000000
...2 : 0.290314 0.289868
...3 : 0.048422 0.048480
...4 : 0.010257 0.010278
...5 : 0.002030 0.002001
...6 : 0.000423 0.000402
...7 : 0.000077 0.000079
...8 : 0.000013 0.000015
...9 : 0.000001 0.000003
...10 : 0.000001 0.000000
Ouf!
Philippe

SNAPSHOTS



SNAPSHOTS



Discrete versus Continuous Mathematics: Transgressing the Boundaries

Philippe FLAJOLET

December 23, 2008

Recent decades have seen a surge of interest in discrete mathematics and *combinatorics*, where what is at stake is the study of properties of finite objects constructed by a finite set of rules. *Structural combinatorics* is the most visible part of the iceberg in Academia, with markedly profound result, such as the theory of minors or the long-sought solution to the perfect graph conjecture. The problem there is to answer questions such as: *Given such and such construction rules, which properties must the corresponding objects* invariably *satisfy?*

Partly pushed by the needs of several branches of science—from computer science to probability theory to bio-informatics to statistical physics—, the past twenty to thirty years have seen the appearance of a large number of studies dedicated to another (at least) equally fundamental question: *Given such and such construction rules, which properties must the corresponding objects satisfy* in an overwhelming proportion of cases*?* In other words, rather than seeking what is *always true*, we want to charaterize what is **almost always** *true*. In this discrete combinatorial world, we want to **measure** things.

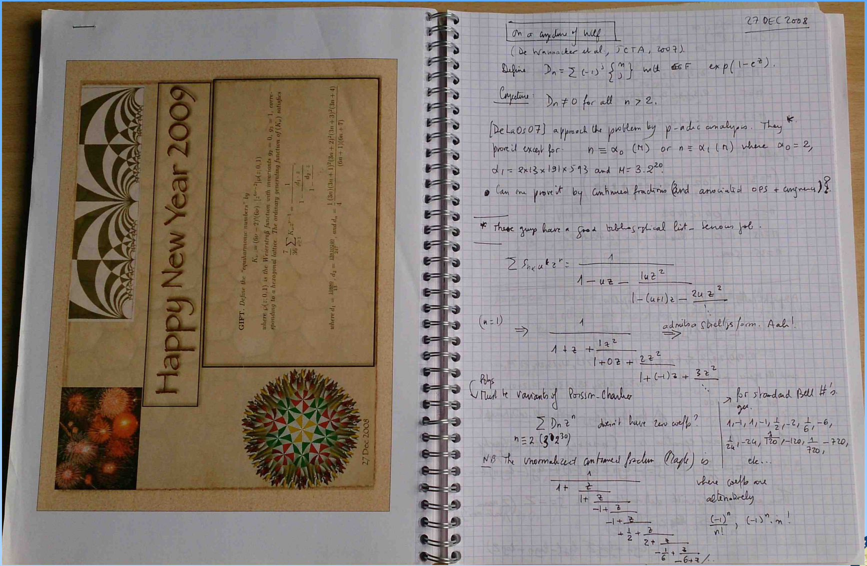
My own research since the early 1980s has been precisely a sustained effort meant to address the characterization of what is "almost always" true amongst the most important structures of mathematics, such as words, trees, mappings, graphs, maps, and permutations. It has constantly alternated between methodological works (e.g., the development of singularity analysis with Odlyzko; the elucidation of the power of the Mellin transform) and research dedicated to solving concrete problems arising from applications (e.g., the coalescence of hashing tables and the universality of Airy laws, the analysis of sequences in relation to probabilistic counting algorithms). An outcome is the publication by Cambridge University Press (at the turn of 2008–2009) of an 825 pages monograph coauthored with Sedgewick and titled *Analytic Combinatorics*, which I will abbreviate as AC[1].

The field of analytic combinatorics as expounded in the book ACconstitutes the basic layer on which the present proposal is built. Roughly, the major theme is that a class of combinatorial structures is reduced to a locally smooth surface (the Riemann surface of a corresponding generating fuction), whose "cracks" (the dignified name is singularities) are seen to contain a host of quantitative information. For instance, as I showed with Odlyzko in 1982 and 1994, the complex-analytic structure of the iteration $y_{h+1} = z + y_h^2$ near

[1] In December 2008, right before the release of AC, qualified by some as *"un best-seller impatiemment attendu"*, the Google Scholar citation score of the manuscript (which had been on the web for several years) was 240, including self-citations: this appears to be substantially *more* than the average citation scoreof an *existing* mathematics book.

1

SNAPSHOTS


Happy New Year 2009
27 Dec 2008
27 DEC 2008
On a conjecture of Wilf

INSIDE LOOK: AGENDA I

- *Stirling #s of the 2nd kind*
- Symbolic method in analysis of tree algorithms
- *Remarks on partitions*
- Trie statistics
- *q-Laguerre polynomials*
- *Simon-Newcomb problem*
- Lexicographic tree height
- Search tree height
- Approximate counting
- Complexity calculus
- Markov chains
- Exp-variate generation
- Talk by Guy Fayolle
- Extension of approximate counting
- Asymptotics/Mellin
- *Distribution of path areas*
- Pippenger's communication protocols
- *Combinatorial sums asymptotics*
- DST asymptotics
- Grid file algorithms
- *Functional graphs*
- *Differential equations & linear systems*

AGENDA I

1982/6/6: after visit to Bell Labs, initiated random mapping statistics with Odlyzko

RANDOM MAPPING STATISTICS

Philippe Flajolet
INRIA Rocquencourt
F–78150 Le Chesnay (France)

Andrew M. Odlyzko
AT&T Bell Laboratories
Murray Hill, NJ 07974 (USA)

Abstract. Random mappings from a finite set into itself are either a heuristic or an exact model for a variety of applications in random number generation, computational number theory, cryptography, and the analysis of algorithms at large. This paper introduces a general framework in which the analysis of about twenty characteristic parameters of random mappings is carried out: These parameters are studied systematically through the use of generating functions and singularity analysis. In particular, an open problem of Knuth is solved, namely that of finding the expected diameter of a random mapping. The same approach is applicable to a larger class of discrete combinatorial models and possibilities of automated analysis using symbolic manipulation systems ("computer algebra") are also briefly discussed.

Initiated in 1982 $\Longrightarrow$ Published in 1990

DEEPER LOOK

Flajolet, Knuth & Pittel (1989)
The first cycles in an evolving graph

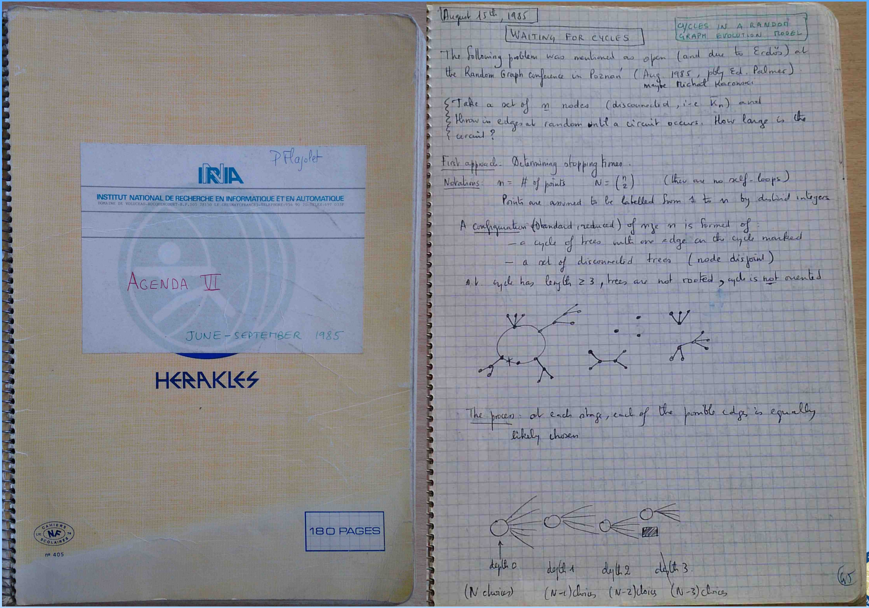
⇓

Backhouse's constant

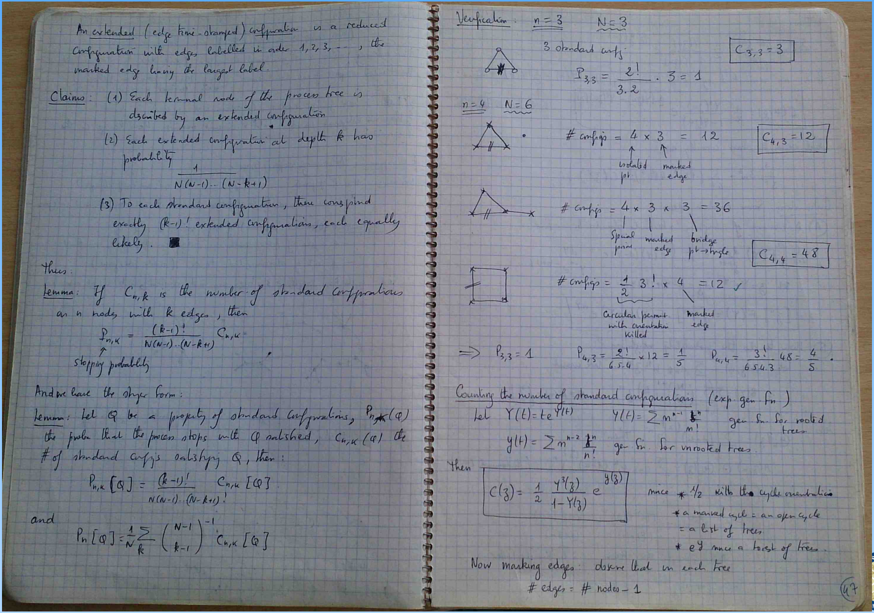
⇓

Gray code function

FIRST CYCLE IN EVOLVING GRAPHS



FIRST CYCLE IN EVOLVING GRAPHS



An extended (edge time-stamped) configuration is a reduced configuration with edges labelled in order 1, 2, 3, ..., the marked edge having the largest label.

Claims: (1) Each terminal node of the process tree is described by an extended configuration

(2) Each extended configuration at depth k has probability

$$\frac{1}{N(N-1)\cdots(N-k+1)}$$

(3) To each standard configuration, there correspond exactly $(k-1)!$ extended configurations, each equally likely. ■

Thus:

Lemma: If $C_{n,k}$ is the number of standard configurations on n nodes with k edges, then

$$P_{n,k} = \frac{(k-1)!}{N(N-1)\cdots(N-k+1)} C_{n,k}$$

↑ stopping probability

And we have the stronger form:

Lemma: Let Q be a property of standard configurations, $P_{n,k}(Q)$ the proba that the process stops with Q satisfied, $C_{n,k}(Q)$ the # of standard configs satisfying Q, then:

$$P_{n,k}[Q] = \frac{(k-1)!}{N(N-1)\cdots(N-k+1)} C_{n,k}[Q]$$

and

$$P_n[Q] = \frac{1}{N}\sum_k \binom{N-1}{k-1}^{-1} C_{n,k}[Q]$$

Verification: $n=3$ $N=3$

3 standard configs $C_{3,3}=3$

$$P_{3,3} = \frac{2!}{3.2}\cdot 3 = 1$$

$n=4$ $N=6$

configs $= 4 \times 3 = 12$ (isolated pt, marked edge) $C_{4,3}=12$

configs $= 4 \times 3 \times 3 = 36$ (special point, marked edge, bridge pt → triangle) $C_{4,4}=48$

configs $= \frac{1}{2} 3! \times 4 = 12$ ✓ (circular permut with orientation killed, marked edge)

$\Longrightarrow$ $P_{3,3}=1$ $P_{4,3} = \frac{2!}{6.5.4}\times 12 = \frac{1}{5}$ $P_{4,4} = \frac{3!}{6.5.4.3}\, 48 = \frac{4}{5}$.

Counting the number of standard configurations (exp. gen. fn)

Let $Y(t) = te^{Y(t)}$ $Y(t) = \sum n^{n-1}\frac{t^n}{n!}$ gen. fn. for rooted trees

$y(t) = \sum n^{n-2}\frac{t^n}{n!}$ gen. fn. for unrooted trees

Then

$$C(z) = \frac{1}{2}\frac{Y^3(z)}{1-Y(z)} e^{y(z)}$$

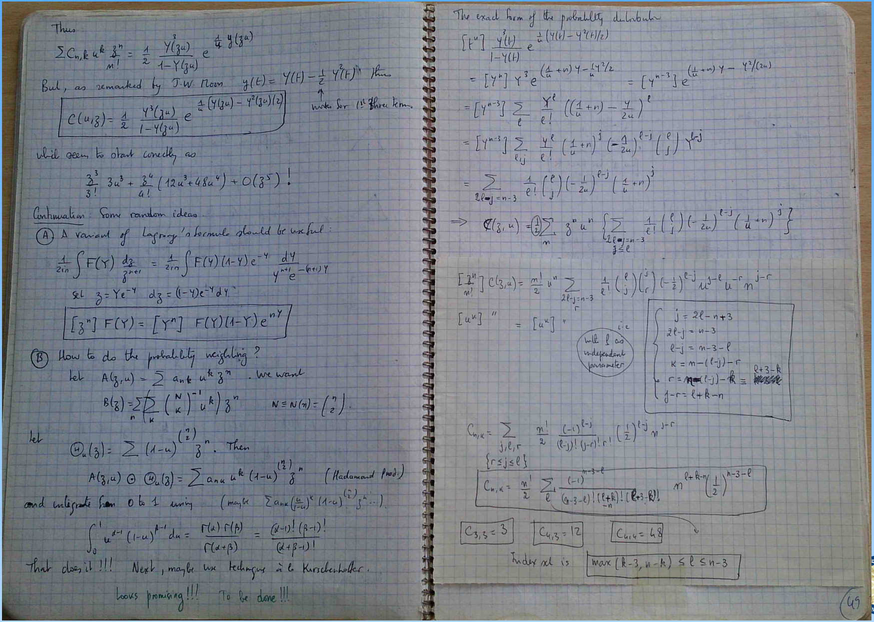
since * ½ kills the cycle orientation
* a marked cycle = an open cycle = a list of trees
* e^y since a forest of trees.

Now marking edges: observe that in each tree

edges = # nodes − 1

(47)

FIRST CYCLE IN EVOLVING GRAPHS



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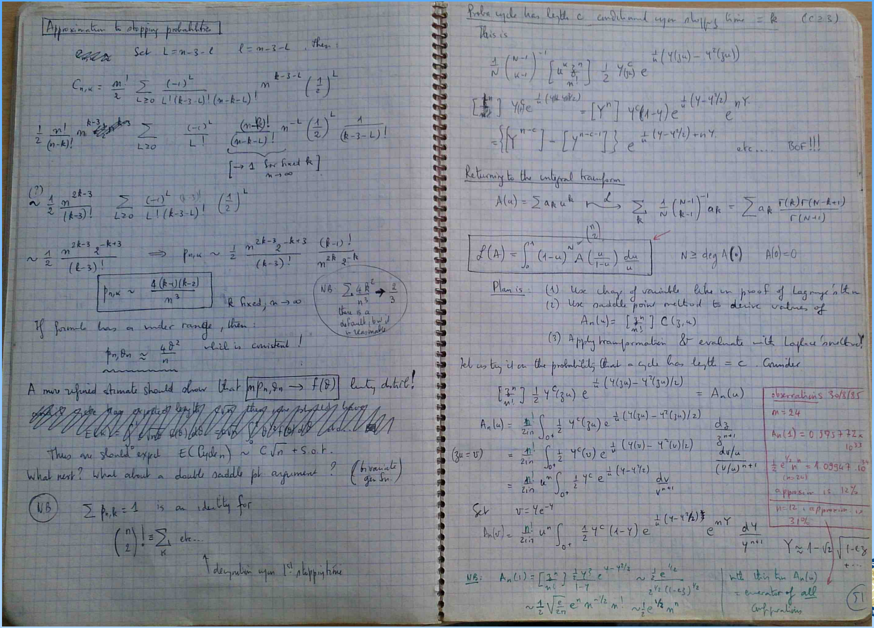

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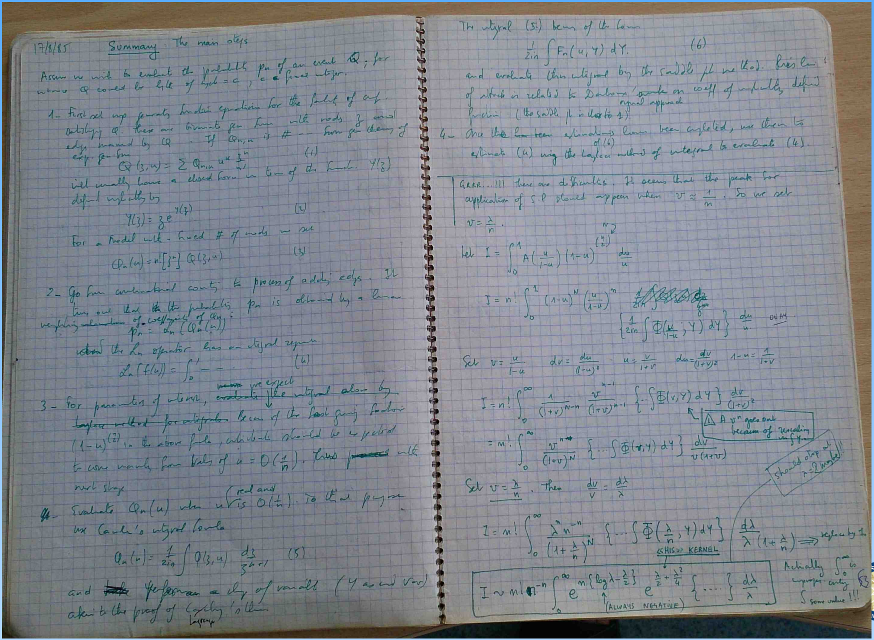

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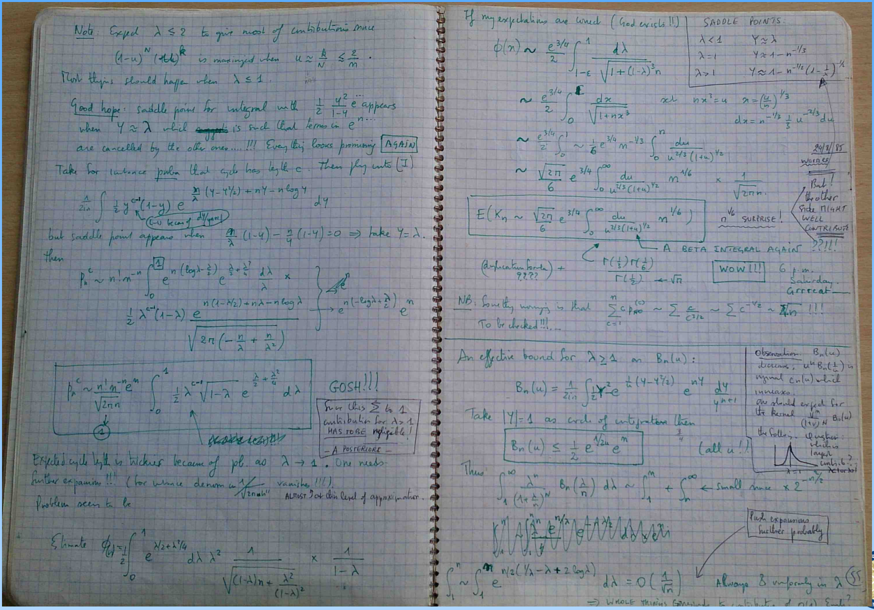

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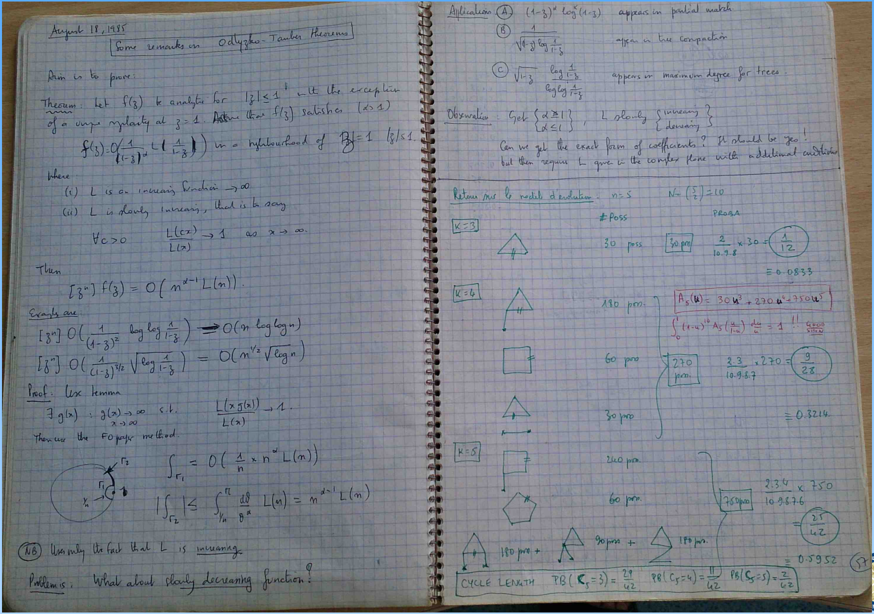

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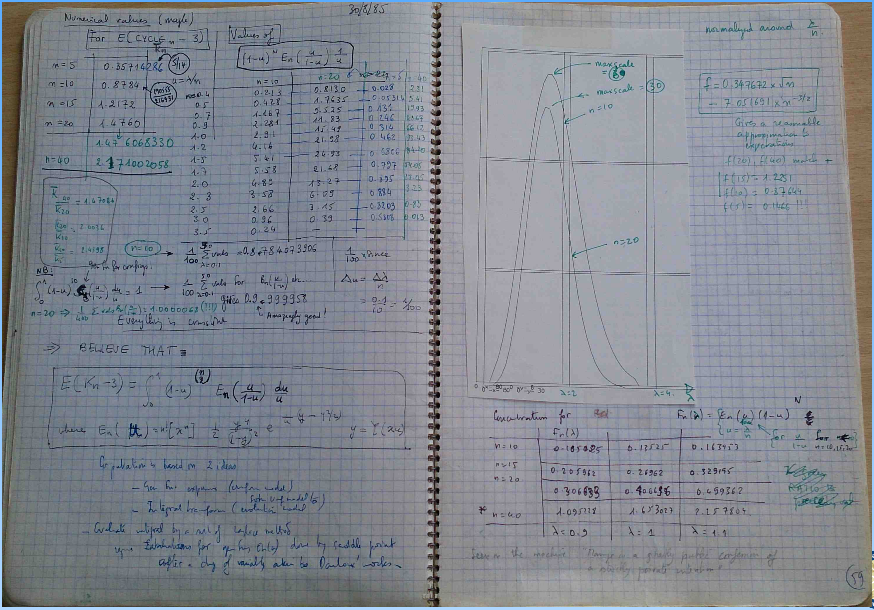
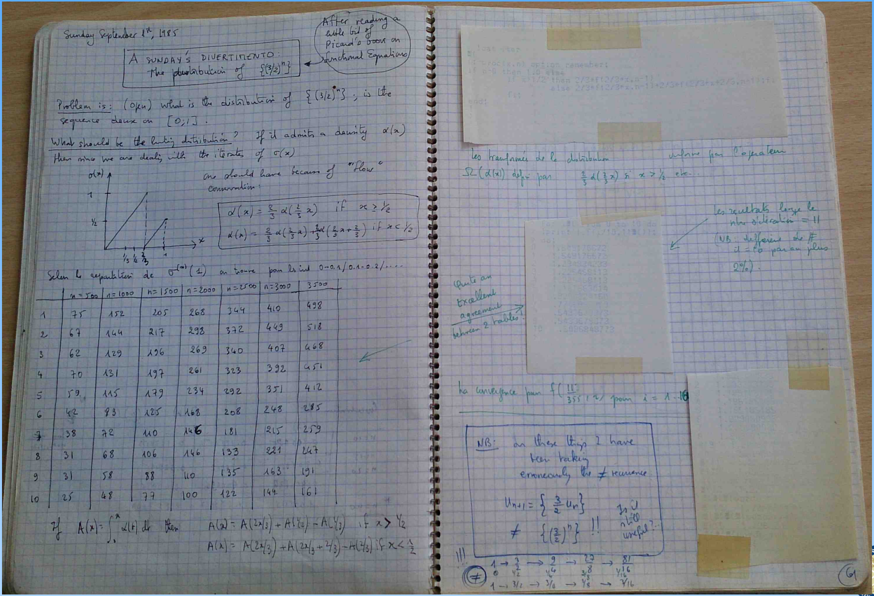


FIRST CYCLE IN EVOLVING GRAPHS



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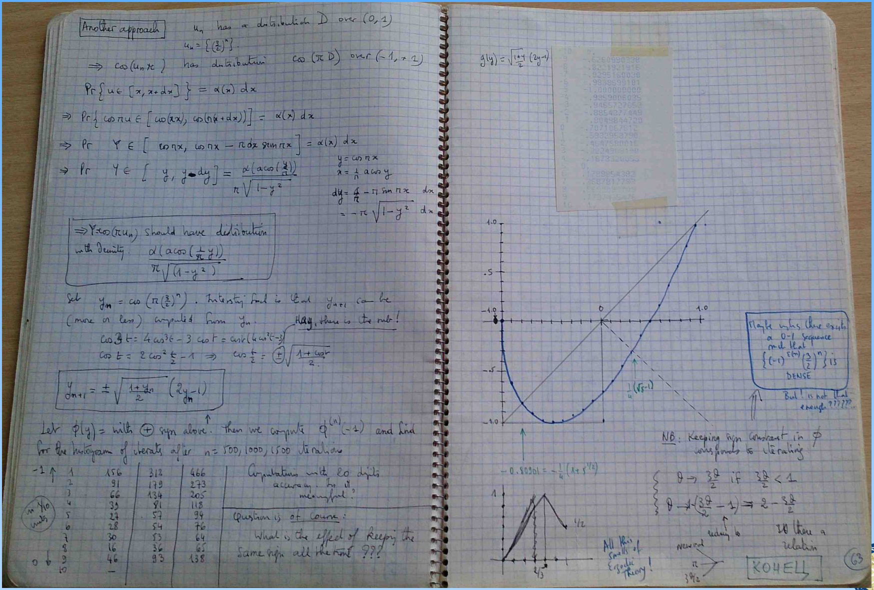

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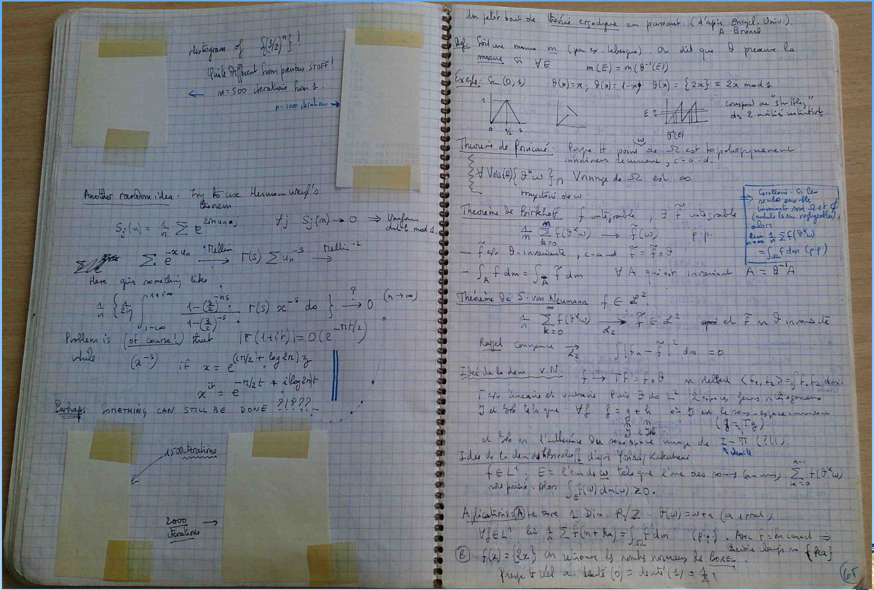

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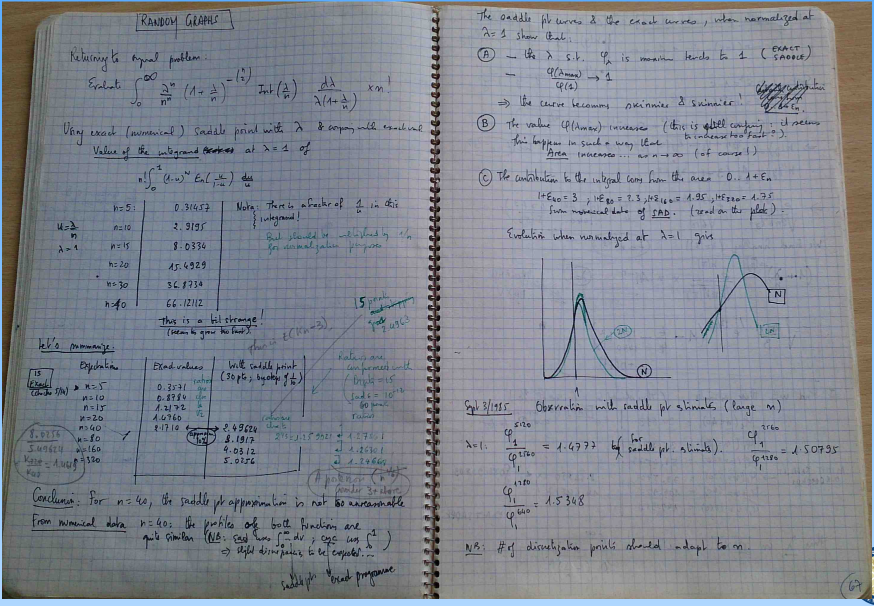

RANDOM GRAPHS

Returning to original problem:

Evaluate $\int_0^{\infty} \frac{\lambda^n}{n^n} \left(1+\frac{\lambda}{n}\right)^{-\binom{n}{2}} \mathrm{Int}\left(\frac{\lambda}{n}\right) \frac{d\lambda}{\lambda(1+\frac{\lambda}{n})} \times n!$

Using exact (numerical) saddle point with λ & comparing with exact value

Value of the integrand at $\lambda = 1$ of

$n!\int_0^1 (1-u)^n E_n\left(\frac{u}{1-u}\right)\frac{du}{u}$

$u = \frac{\lambda}{n}$, $\lambda = 1$

n=5	0.31457
n=10	2.9195
n=15	8.0334
n=20	15.4929
n=30	36.8734
n=40	66.12112

Nota: There is a factor of $\frac{1}{u}$ in this integrand!

But should be multiplied by 1/n for normalization purposes

This is a bit strange! (seems to grow too fast).

Let's summarize.

Expectations	Exact values	With saddle point (30 pts; by steps of $\frac{1}{10}$)
n=5	0.3571	
n=10	0.8784	
n=15	1.2172	
n=20	1.4760	
n=40	2.1710	2.49624
n=80		3.1917
n=160		4.0312
n=320		5.0256

Conclusion: For n=40, the saddle pt approximation is not too unreasonable

From numerical data n=40: the profiles of both functions are quite similar (NB: sad uses $\int_0^{\infty} dv$; exc uses $\int_0^1$)

⇒ slight discrepancies to be expected.

The saddle pt curves & the exact curves, when normalized at $\lambda = 1$ show that:

(A) — the λ s.t. φ_λ is maximum tends to 1 (EXACT SADDLE)

— $\frac{\varphi(\lambda_{max})}{\varphi(1)} \to 1$

⇒ the curve becomes skinnier & skinnier!

(B) The value $\varphi(\lambda_{max})$ increases (this is still confusing: it seems to increase too fast?). This happens in such a way that Area increases... as $n \to \infty$ (of course!)

(C) The contribution to the integral comes from the area $0..1+\varepsilon_n$

$1+\varepsilon_{40} = 3$; $1+\varepsilon_{80} = 2.3$; $1+\varepsilon_{160} = 1.95$; $1+\varepsilon_{320} = 1.75$ from numerical data of SAD. (read on the plot).

Evolution when normalized at $\lambda = 1$ gives

Sept 3/1985 Observation with saddle pt estimates (large n)

$\lambda = 1$: $\frac{\varphi_1^{5120}}{\varphi_1^{2560}} = 1.4777$ (for saddle pt. estimates). $\frac{\varphi_1^{2560}}{\varphi_1^{1280}} = 1.50795$

$\frac{\varphi_1^{1280}}{\varphi_1^{640}} = 1.5348$

NB: # of discretization points should adapt to n.

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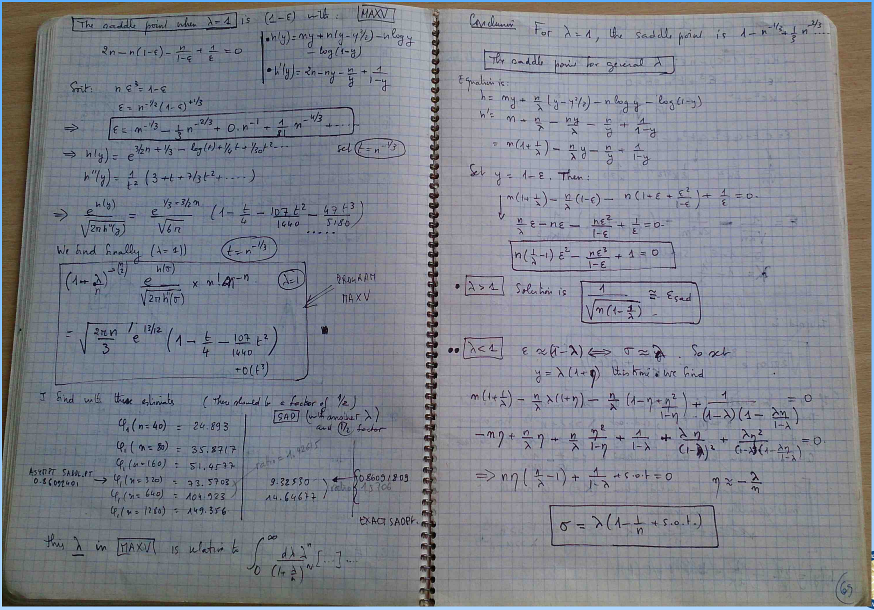
The saddle point when λ = 1 is (1 − ε) with: MAXV
Conclusion: For λ = 1, the saddle point is 1 − n^{-1/3} + 1/3 n^{-2/3} ...
The saddle point for general λ
Equation is:
Set y = 1 − ε. Then:
λ > 1 Solution is
λ < 1
We find finally (λ = 1)
PROGRAM MAXV
I find with these estimates
ASYMPT SADDL.PT
EXACT SADPT.
this λ in MAXV is relative to
σ = λ(1 − 1/n + s.o.t.)
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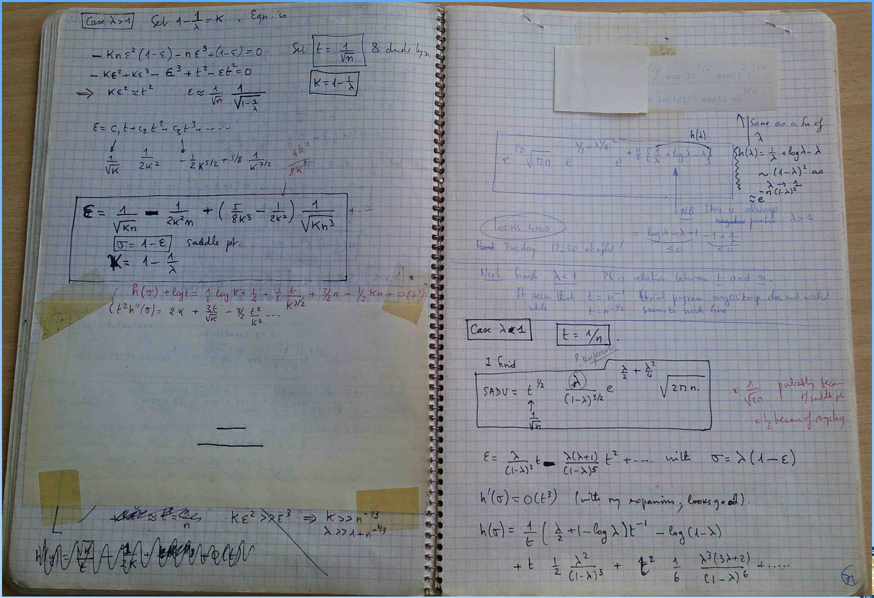
FIRST CYCLE IN EVOLVING GRAPHS



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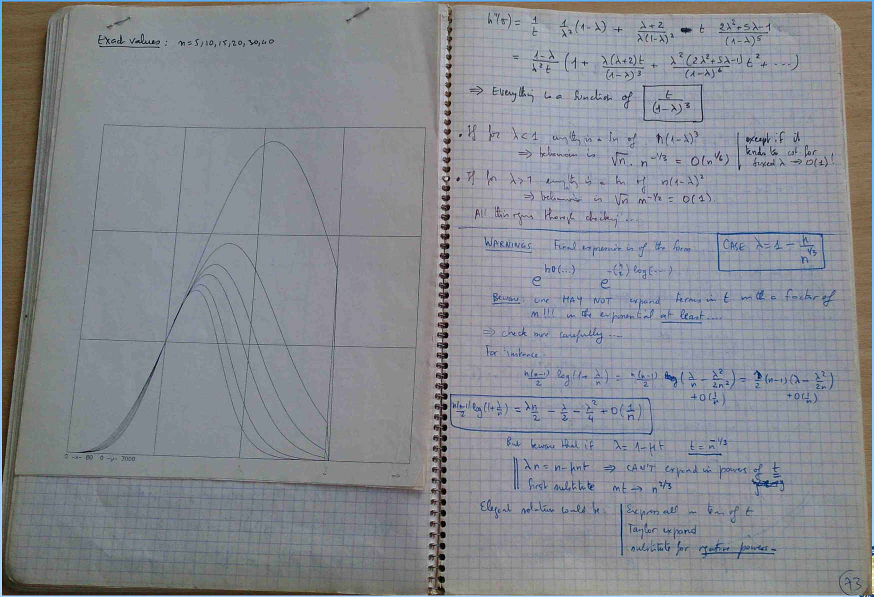
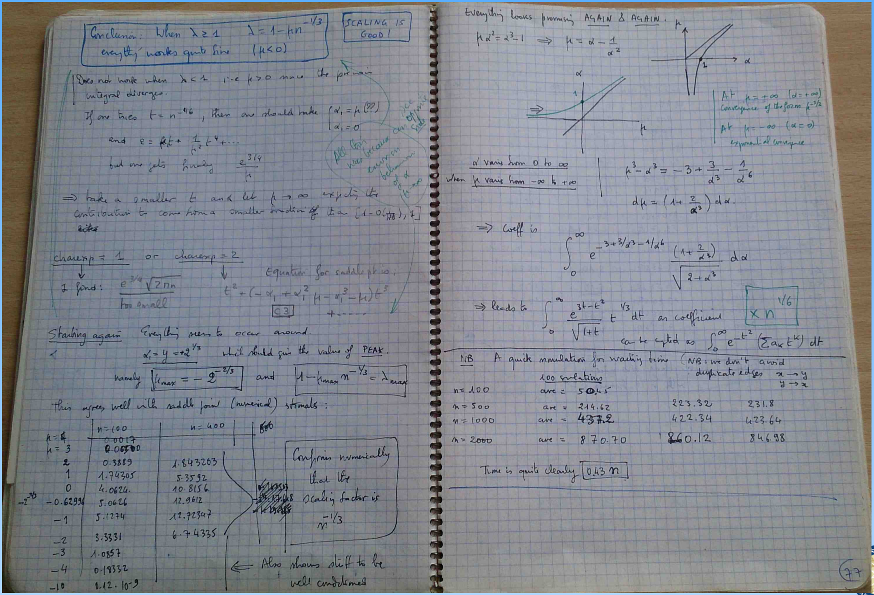


FIRST CYCLE IN EVOLVING GRAPHS

FIRST CYCLE IN EVOLVING GRAPHS



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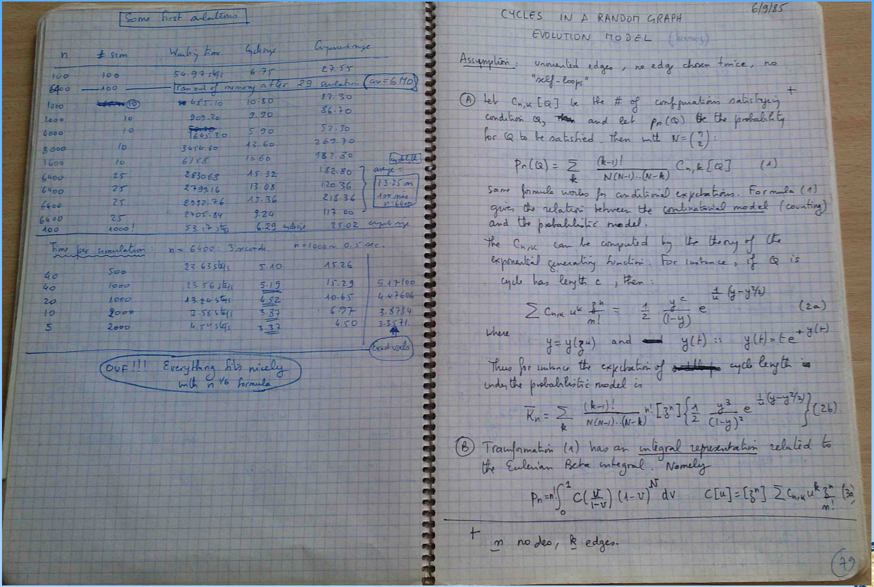

Some first simulations

n	# sim	Waiting time	Cycle size	Component size
100	100	54.97 steps	6.75	27.55
~~6~~6400	100	ran out of memory after 29 simulation (av = 6 MO)		
1000	~~[illegible]~~ (10)	~~[illegible]~~ 485.10	10.80	87.30
2000	10	909.20	9.90	86.70
4000	10	~~[illegible]~~ 1645.30	5.90	52.30
8000	10	3454.60	13.60	267.70
1600	10	6158	1a 60	182.80
6400	25	283068	15.32	182.80
6400	25	279916	13.08	120.36
6400	25	2992.76	15.36	218.36
6400	25	2705.84	9.24	117.00
100	1000!	53.17 steps	6.29 cycle size	25.02 component size

cycle length average = 13.25 on 100 runs n = 6400

Time per simulation: n = 6400: 3 seconds   n = 1000 ~ 0.5 sec.

40	500	23.63 steps	5.10	15.26	
40	1000	23.56 steps	5.19	15.29	5.17100
20	1000	13.24 steps	4.52	10.65	4.47606
10	2000	7.58 steps	3.87	6.97	3.8784
5	2000	4.54 steps	3.37	4.50	3.3571.

Exact vals

OUF!!! Everything fits nicely with $n^{1/6}$ formula

6/9/85

CYCLES IN A RANDOM GRAPH

EVOLUTION MODEL (summary)

Assumption: unoriented edges, no edge chosen twice, no "self-loops"

(A) Let $C_{n,k}[Q]$ be the # of configurations[+] satisfying condition Q, ~~then~~ and let $p_n(Q)$ be the probability for Q to be satisfied. Then with $N = \binom{n}{2}$:

$$p_n(Q) = \sum_k \frac{(k-1)!}{N(N-1)\cdots(N-k)} C_{n,k}[Q] \qquad (1)$$

Same formula works for conditional expectations. Formula (1) gives the relation between the combinatorial model (counting) and the probabilistic model.

The $C_{n,k}$ can be computed by the theory of the exponential generating function. For instance, if Q is cycle has length c, then:

$$\sum C_{n,k} u^k \frac{z^n}{n!} = \frac{1}{2} \frac{y^c}{(1-y)} e^{\frac{1}{u}(y - y^2/2)} \qquad (2a)$$

where $y = y(zu)$ and ~~[illegible]~~ $y(t) :: y(t) = t e^{+y(t)}$

Thus for instance the expectation of ~~[illegible]~~ cycle length ~~is~~ under the probabilistic model is

$$\overline{K}_n = \sum_k \frac{(k-1)!}{N(N-1)\cdots(N-k)} n! [z^n] \left\{ \frac{1}{2} \frac{y^3}{(1-y)^2} e^{\frac{1}{u}(y - y^2/2)} \right\} \qquad (2b)$$

(B) Transformation (1) has an integral representation related to the Eulerian Beta integral. Namely

$$p_n = n! \int_0^1 C\left(\frac{v}{1-v}\right)(1-v)^N dv \qquad C[u] = [z^n] \sum C_{n,k} u^k \frac{z^n}{n!} \qquad (3a)$$

[+] n nodes, k edges.

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FIRST CYCLE IN EVOLVING GRAPHS

which becomes.

$$p_n = n! \int_0^\infty \frac{v^n}{(1+v)^N}\left\{[z^n]\, C(z,\tfrac{1}{v})\right\} \frac{dv}{v(1+v)} \qquad (3b)$$

and setting $v = \lambda/n$:

$$p_n = n!\, n^{-n} \int_0^\infty \frac{\lambda^n}{(1+\frac{\lambda}{n})^N}\left\{[z^n]\, C(z,\tfrac{\lambda}{n})\right\} \frac{d\lambda}{\lambda(1+\frac{\lambda}{n})} \qquad (3c)$$

(C) The idea is then to evaluate $C(z,\frac{\lambda}{n})$ numerically for fixed λ (actually only $\lambda = O(1)$ matters somewhat) by

(C1) The Cauchy integral (of course!)

(C2) the change of variable of the proof of Lagrange Inversion theorem

(C3) Finally saddle point method.

For instance if we take as "parameter" Q the expectation of cycle length (=3), by (C1) (4)

$$K_n = n!\, n^{-n}\int_0^\infty \frac{\lambda^n}{(1+\frac{\lambda}{n})^N}\left\{\frac{1}{2i\pi}\int \frac{1}{2}\frac{y^3}{(1-y)^2}\, e^{\frac{n}{\lambda}(y-y^2/2)}\frac{dz}{z^{n+1}}\right\}\frac{d\lambda}{\lambda(1+\frac{\lambda}{n})}, \qquad y = z e^y$$

Then by C2 (5)

$$K_n = n!\, n^{-n}\int_0^\infty \frac{\lambda^n}{(1+\frac{\lambda}{n})^N}\left\{\frac{1}{2i\pi}\int \frac{1}{2}\frac{y^2}{(1-y)^2}\, e^{\frac{n}{\lambda}(y-y^2)+ny-n\log y}\,dy\right\}\frac{d\lambda}{\lambda(1+\frac{\lambda}{n})}$$

and applying (C3) represents some challenge.

(D) The saddle point method

What happens ultimately is the integrand of (5) (ie the big function of λ and n in $\int_0^\infty \ldots d\lambda$) has a peak at $\lambda = 1 + \varepsilon_n$.

For fixed λ, 2 cases appear

$\lambda < 1$, $\lambda > 1$.

The saddle point $\sigma = \sigma(\lambda, n)$ satisfies an algebraic equation of degree 3 but the ~~one~~ (BRANCH) to be picked changes suddenly at $\lambda = 1$.

For $\lambda < 1$ $\sigma = \lambda + o(1)$.

For $\lambda > 1$ $\sigma = 1 - o(1)$

and using symbolic manipulation systems, I find with $I_n(\lambda)$ denoting the integrand of 5 normalized by $n!\,n^{-n}$:

$\lambda < 1$: $I_n(\lambda) \sim \frac{[illegible]\,\lambda}{(1-\lambda)^{3/2}}\, e^{\lambda/2+\lambda^2/4}$ = $O(1)$ for fixed λ, $n\to\infty$

$\lambda > 1$: $I_n(\lambda) \sim e^{1/2}\sqrt{\frac{n}{2}}\, e^{\lambda/2+\lambda^2/4}\, e^{\frac{n}{2}(\frac{1}{\lambda}+\log\lambda-\lambda)}$ = exponentially small for fixed λ, $n\to\infty$

Thus the contribution should be localized around $\lambda = 1$.

At $\lambda = 1$: $I_n(1) \sim \frac{1}{2}\sqrt{\frac{n}{3}}\, e^{13/12}$ = increases as $O(\sqrt{n})$.

thus the problem is to find how things behave for λ very close to 1 but a function of n itself.

(E) It takes a little while to find that the proper scaling factor is $t = n^{-1/3}$ and we consider now ~~the saddle point~~ $\lambda = 1 - \mu t = 1 - \mu n^{-1/3}$, μ fixed.

Then the saddle point $\sigma = \sigma(\lambda, n)$ has an asymptotic expansion for $t \to 0$ $(n\to\infty)$:

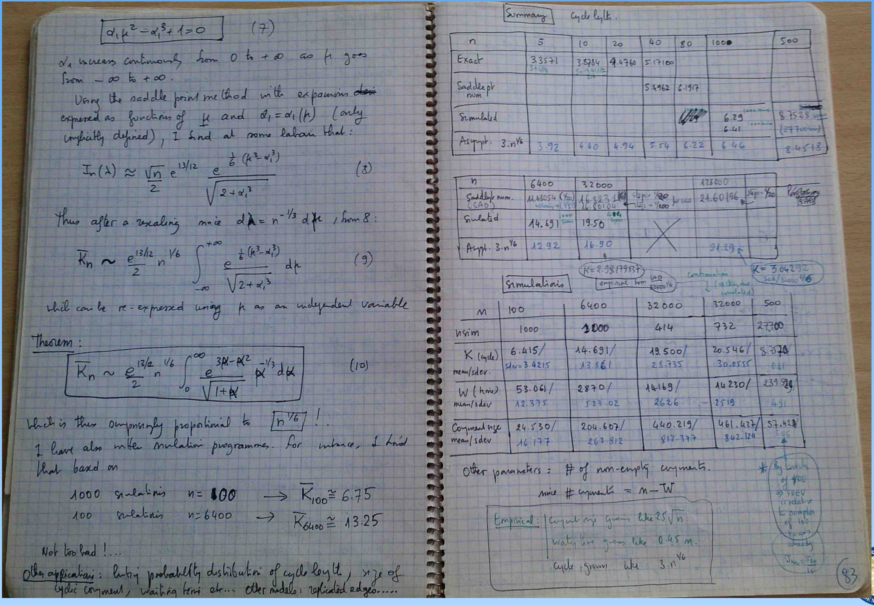
$$\sigma = 1 - \alpha_1(t) + \frac{\mu + \alpha_1(\mu)}{(2\mu - 3\alpha_1(\mu))\,\alpha_1(\mu)} \times t^2 + O(t^3) \ldots \qquad (6)$$

(more terms are necessary and have been computed using maple).

In (6) $\alpha_1(\mu)$ is an algebraic function of degree 3 defined by

(81)

FIRST CYCLE IN EVOLVING GRAPHS



$\alpha_1 \mu^2 - \alpha_1^3 + 1 = 0$ (7)

α_1 increases continuously from 0 to $+\infty$ as μ goes from $-\infty$ to $+\infty$.

Using the saddle point method with expansions expressed as functions of μ and $\alpha_1 = \alpha_1(\mu)$ (only implicitly defined), I find at some labour that:

$$I_n(\lambda) \approx \frac{\sqrt{n}}{2} e^{13/12} \frac{e^{\frac{1}{6}(\mu^3 - \alpha_1^3)}}{\sqrt{2+\alpha_1^3}} \quad (8)$$

Thus after a rescaling since $d\lambda = n^{-1/3} d\mu$, from 8:

$$\overline{K}_n \sim \frac{e^{13/12}}{2} n^{1/6} \int_{-\infty}^{+\infty} \frac{e^{\frac{1}{6}(\mu^3-\alpha_1^3)}}{\sqrt{2+\alpha_1^3}} d\mu \quad (9)$$

which can be re-expressed using α as an independent variable

Theorem:

$$\overline{K}_n \sim \frac{e^{13/12}}{2} n^{1/6} \int_0^{\infty} \frac{e^{3\alpha - \alpha^2}}{\sqrt{1+\alpha}} \alpha^{-1/3} d\alpha \quad (10)$$

which is thus surprisingly proportional to $n^{1/6}$!.

I have also written simulation programmes. For instance, I find that based on

1000 simulations $n = 100 \longrightarrow \overline{K}_{100} \cong 6.75$

100 simulations $n = 6400 \longrightarrow \overline{K}_{6400} \cong 13.25$

Not too bad !...

Other applications: limiting probability distribution of cycle length, size of cyclic component, waiting time etc... Other models: replicated edges....

Summary — Cycle length.

n	5	10	20	40	80	1000	500
Exact	3.3571	3.8784	4.4760	5.17100			
Saddle pt num				5.4962	6.1917		
Simulated						6.29 / 6.41	8.7528 (27700 sims)
Asympt. $3.n^{1/6}$	3.92	4.40	4.94	5.54	6.22	6.46	8.4518

n	6400	32000		128000	
Saddle pt num. (SAD)	11.48054 (1/20)	16.8231 / 16.80104	steps = 1/20, steps = 1/100 for SAD	21.60196	steps = 1/20
Simulated	14.691 (1000 sims)	19.56 (414 sims)	X		
Asympt. $3.n^{1/6}$	12.92	16.90		21.29	

K = 2.98179537 (empirical from SAD / $32000^{1/6}$)

K = 3.04292 (SAD / $32000^{1/6}$)

Simulations

n	100	6400	32000	32000	500
nsim	1000	1000	414	732	27700
K (cycle) mean/sdev.	6.415 / sdev = 3.4215	14.691 / 13.861	19.500 / 28.735	20.546 / 30.0555	8.7570 / 6.61
W (time) mean/sdev	53.061 / 12.375	2870 / 523.02	14169 / 2626	14230 / 2519	239.529 / 4.91
Component size mean/sdev	24.530 / 16.177	204.607 / 267.812	440.219 / 812.377	461.427 / 842.124	57.427 / 5.05

Other parameters: # of non-empty components.

since # components = n − W

Empirical: component size grows like $2.5\sqrt{n}$

waiting time grows like $0.45\,n$.

cycle grows like $3.n^{1/6}$

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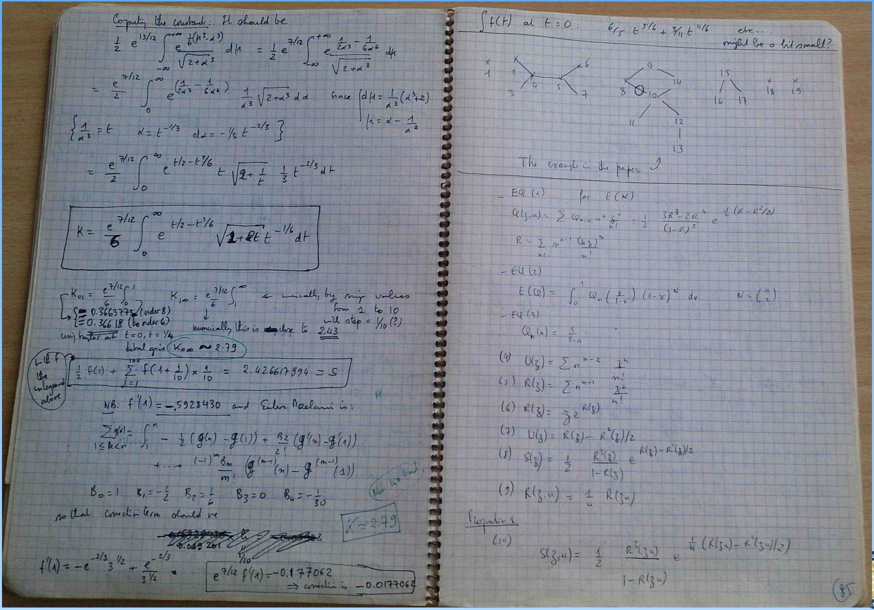
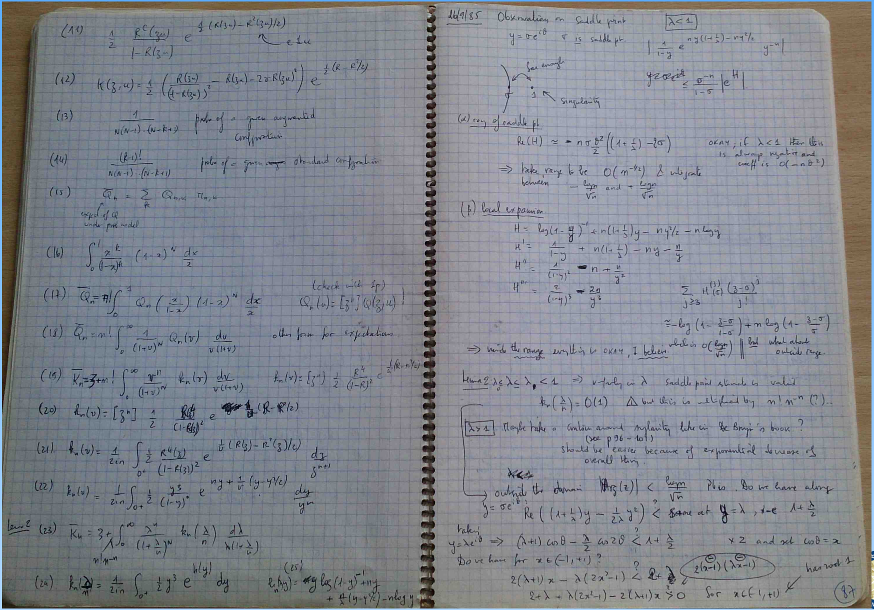
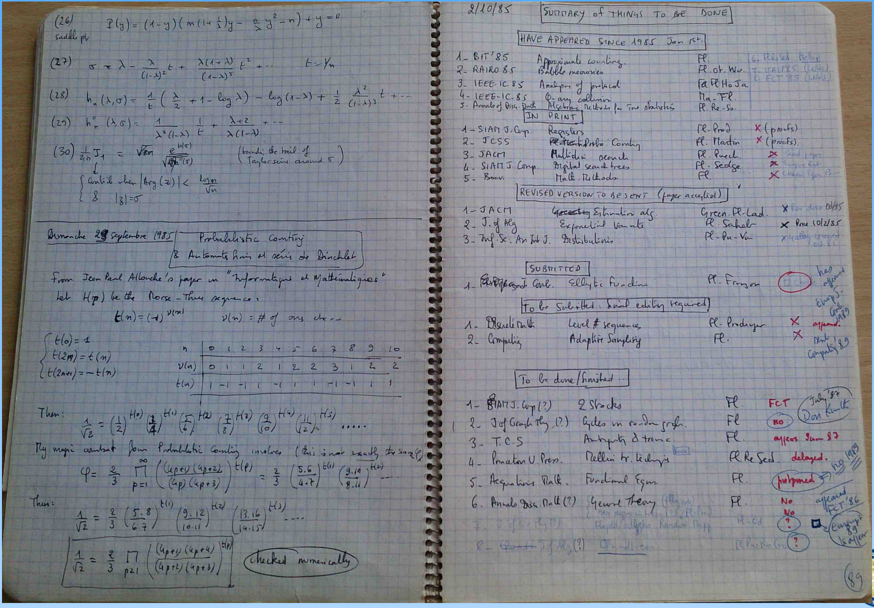
FIRST CYCLE IN EVOLVING GRAPHS



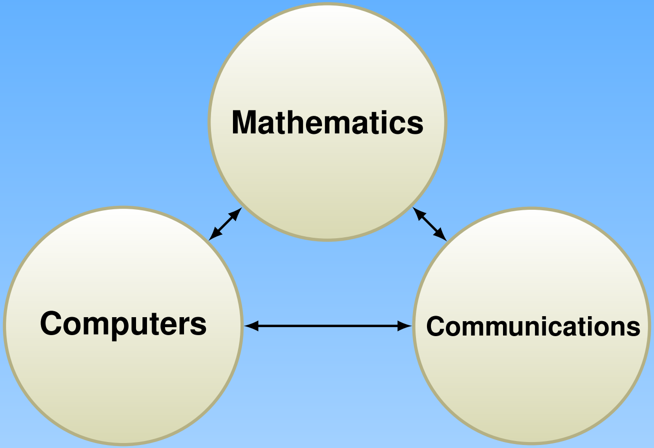
FIRST CYCLE IN EVOLVING GRAPHS



FIRST CYCLE IN EVOLVING GRAPHS



SYNERGISTIC INTERACTION



BACKHOUSE'S CONSTANT

Radius of cv of

$$\frac{1}{1+\sum_{k\geqslant 1} p_k z^k}$$

A030018 Coefficients in 1/(1+P(x)), where P(x) is the generating function of the primes. 8

1, -2, 1, -1, 2, -3, 7, -10, 13, -21, 26, -33, 53, -80, 127, -193, 254, -355, 527, -764, 1149, -1699, 2436, -3563, 5133, -7352, 10819, -15863, 23162, -33887, 48969, -70936, 103571, -150715, 219844, -320973, 466641, -679232, 988627, -1437185, 2094446, -3052743 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS a(n+1)/a(n) => ~-1.4560749485826896714. - Zak Seidov, Oct 01 2011.

Backhouse's constant

From Wikipedia, the free encyclopedia

Backhouse's constant is a mathematical constant founded by N. Backhouse and is approximately 1.456 074 948.

It is defined by using the power series such that the coefficients of successive terms are the prime numbers:

$$P(x) = 1 + \sum_{k=1}^{\infty} p_k x^k = 1 + 2x + 3x^2 + 5x^3 + 7x^4 + \cdots$$

and where

$$Q(x) = \frac{1}{P(x)} = \sum_{k=0}^{\infty} q_k x^k.$$

Then:

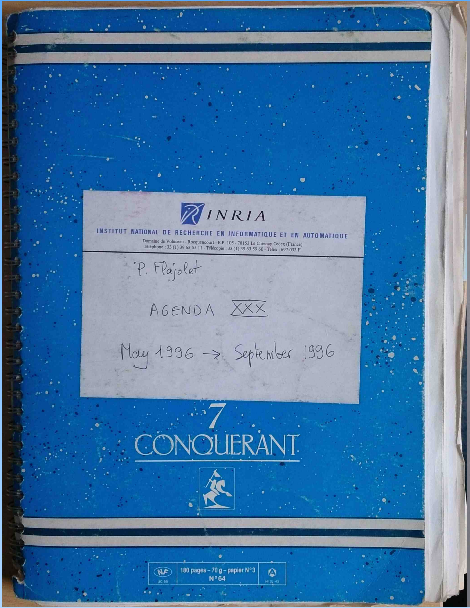
$$\lim_{k\to\infty}\left|\frac{q_{k+1}}{q_k}\right| = 1.45607\ldots$$ (sequence A072508 in OEIS).

The limit was conjectured to exist by Backhouse which was later proved by P. Flajolet.

Binary	1.01110100110000010101001111101100···
Decimal	1.45607494858268967139959535111654···
Hexadecimal	1.74C153ECB002353B12A0E476D3ADD···
Continued fraction	$1+\cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{5+\cfrac{1}{4+\ddots}}}}$

Note that this continued fraction is not periodic.

BACKHOUSE'S CONSTANT



Backhouse's Constant

Let P(x) be the formal power series whose nth term has coefficient equal to the nth prime number:

$$P(x) = \sum_{k=0}^{\infty} p_k x^k = 1 + 2\cdot x + 3\cdot x^2 + 5\cdot x^3 + 7\cdot x^4 + 11\cdot x^5 + 13\cdot x^6 + \ldots$$

Let Q(x) be the formal power series defined by

$$P(x)\cdot Q(x) = 1$$

Thus Q(x) is the formal reciprocal of P(x) as a power series. Observe that this is pure formal algebra: no questions of analytical convergence are involved at all.

Q(x) is an alternating series whose coefficients q_n are monotonically increasing in magnitude. Nigel Backhouse has observed that the ratios of successive coefficients tend to a certain constant, i.e., it appears that

$$\lim_{n\to\infty} \left|\frac{q_{n+1}}{q_n}\right| = 1.45607494858268967139959535111654356\ldots$$

In a personal communication, Backhouse wrote:

```
The approximation given was generated in 37 seconds using Maple V
(release 3) in batch mode on a Silicon Graphics Irix6.  P(x) was
taken to 550 terms and Q(x) produced as the Taylor series of P(x)^(-1).

Unfortunately, I have no references to this result or anything
like it. In particular I have no evidence as to the originality
of my observation. I was just curious, as someone with an amateur
interest in number theory !

I should, of course, be very interested to hear, if, as a result
of your enterprise, someone has anything to add to my rather thin
story.
```

The 35-place decimal approximation above also appears at the CECM Inverse Symbolic Calculator web site. I am grateful to Simon Plouffe for pointing out to me the existence of this constant and to Nigel Backhouse for providing the information on which this essay is based.

Relevant Mathcad files will be included as time permits.

Return to the Favorite Mathematical Constants page .
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This page was updated November 21, 1995

BACKHOUSE CONSTANT

ON THE EXISTENCE AND THE COMPUTATION OF
BACKHOUSE'S CONSTANT

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November 25, 1995
<Philippe.Flajolet@inria.fr>

I. THE PROBLEM

Let p(n) be the n-th prime, with p(1)=2, and define

$$P(z) = 1 + \sum_{n=1}^{\infty} p(n)\, z^n .$$

Nigel Backhouse examines the coefficients q(n) in the series Q(z)=1/P(z):

$$Q(z) = \frac{1}{P(z)} = \sum_{n=0}^{\infty} q(n)\, z^n .$$

He notices empirically that the q(n) alternate in sign and that the ratio between successive values tends a constant equal (up to sign) to 1.45607... and called now ''Backhouse's constant''. See the description in Steven Finch's pages
<http://www.mathsoft.com/asolve/constant/backhous/backhous.html>.

II. ANALYSIS

Here is what goes on. By the Prime Number Theorem, we have p(n)~nlog(n), and at any rate p(n)<(n+1)^2 for all n. Thus, P(z) is an analytic function in |z|<1. Accordingly, Q(z) is meromorphic in |z|<1 and has only finitely many poles in any subdisk |z|<=1-eps of the unit disk. Since P(0)=1, Q(z) is analytic at 0. Thus, by Cauchy's coefficient formula,

$$q(n) := \left(\frac{1}{2\, i\, pi}\right) \int \frac{Q(z)}{z^{(n+1)}}\, dz$$

where the integration contour is a sufficiently small circle around 0. We observe that P(z) has a unique zero at s0=-0.686 inside the disk of radius 0.75. Thus, integrating along |z|=0.75 and taking into account the residue of Q(z) at z=s0 gives us

$$q(n) = \left(\frac{1}{s0\, P'(s0)}\right) a0^{n} + O(.75^{(-n)})$$

where a0=1/s0=-1.45607 is Backhouse's constant. This formula is quite good as its error term is of rate 1/0.75=1.33, hence exponentially smaller than the dominant term.

It is possible to go farther by fishing for the next poles. In this way one can find better and better asymptotic expansions of the type

$$q(n) = c[0]\, a[0]^{n} + c[1]\, a[1]^{n} + c[2]\, a[2]^{n} + etc$$

(with suitable modifications if multiple poles were to be encountered), where a[i]=1/s[i] and s[i] is the i-th zero of P(z).
There doesn't seem to be real poles apart from s0

nor multiple poles, but I have naturally no proof for this observation. Complex conjugate pairs of poles will make the correction terms fluctuating, as usual.

Note that for any collection of zeros given by a numerical process, a corresponding computer-assisted proof could be built. It suffices to use the principle of the argument [Henrici, Complex and Computational Analysis] to make sure that all zeros in a certain disk have indeed been captured. Also, all this proves that the coefficients q(n) eventually alternate in sign. This could be extended to all values of n by using constructive bounds in Cauchy's remainder integral and checking exhaustively the few dozen initial values not covered by the bounds.

Finally, there is a general theorem of Polya-Carlson to the effect that a nonrational function with integer coefficients and radius of convergence 1 admits the unit circle has a natural boundary. Thus, as anticipated, P(z) and Q(z) both have the unit circle as a singular line.

III. A GENERAL REMARK

What we just did is an instance of a general process well known in the analysis of coefficients of meromorphic functions. It is related to methods for coefficient asymptotics, like Darboux's method or singularity analysis, that are especially useful in ''analytic combinatorics''. An example that is close and that I like to use in teaching coefficient asymptotics is the following.

> A composition of an integer n is a sequence of integers >0 that add up to n. The number of compositions of n is 2^(n-1). Now consider compositions whose parts are restricted to be prime numbers 2,3,5,7,11,...
> How many are there?
> Answer: about 0.303655263*1.476228783^n.
> Proof. Work with the series S(z)=1/(1-R(z))
> where R(z)=z^2+z^3+z^5+z^7+z^11+...

Philosophy: This discussion shows these questions to be infinitely easier than true arithmetical ones since this type of problem is (exponentially) oblivious of the fine structure of intervening analytic functions. For instance, one could define the ''twin-Backhouse'' constant by restricting P to terms corresponding to twin prime pairs!! Even though we don't know a lot, we could still PROVE existence of the asymptotic form and COMPUTE the new constant to 1,000 digits in a matter of minutes. Similarly for the Fermat-Backhouse and the Mersenne-Backhouse constants!!!

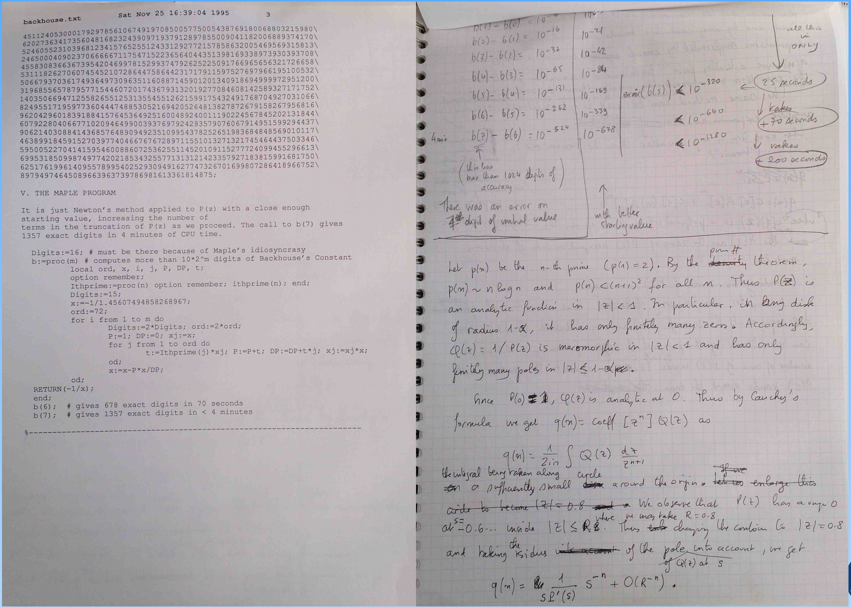
Advertisement: A tutorial on these questions (''Complex Asymptotics and Generating Functions'', INRIA Tech. Rep. 2026, Sept 1993) is available from <Philippe.Flajolet@inria.fr> and is going to be part of a forthcoming book by P. Flajolet and R. Sedgewick entitled ''Analytic Combinatorics''.

IV. NUMERICAL VALUE OF BACKHOUSE'S CONSTANT

Here is in connection with Simon Plouffe's dictionary of real numbers
http://www.cecm.sfu.ca/projects/ISC.html
the value of Backhouse's constant to some 1300 Digits
(determined in 4 minutes of CPU time with a Maple V.3 program on a DEC Alpha 3000 station.)

1.456074948582689671399595351116543557653178374847131540270702 4\
37414001506265389895599645319401860309109925143619634713548607 7\
51649131212314292035177012831740536952749988025486923070580852 8\

BACKHOUSE CONSTANT



backhouse.txt   Sat Nov 25 16:39:04 1995   3

```
4511240530001792978561067491970850057750054387691800688032159 80\
620273634173560481682324390971937912897855009041182006889374170\
524605523103968123415765255124331292772157858632005469569315813\
246500040902370666667117547152236564044351398169338973930393708\
455830836636739542046997815299374792625225091766965656321726658\
531118262706074545210728644758644231717911597527697966195100532\
506679370361749364973096351160887145901201340918694999972951200\
319685565787957715446072017436793132019277084608142589327171752\
140350669471255826551253135545512621599175432491768704927031066\
824955171959773604447488530521694205264813827872679158267956816\
962042960183918841576453649251600489240011190224567845202131844\
607922804066771020946499003937697924293579076067914951599294437\
906214030884143685764890949235109954378252651983684848569010117\
463899184591527039774046676767289711551013271321745464437503346\
595005227041415954600886072536255114520109115277724099455296613\
699531850998749774202185343255771313121423357927183815991681750\
625176199614095578995402529309491627747326701699807286418966752\
8979497464508966396373978698161336181 4875;
```

V. THE MAPLE PROGRAM

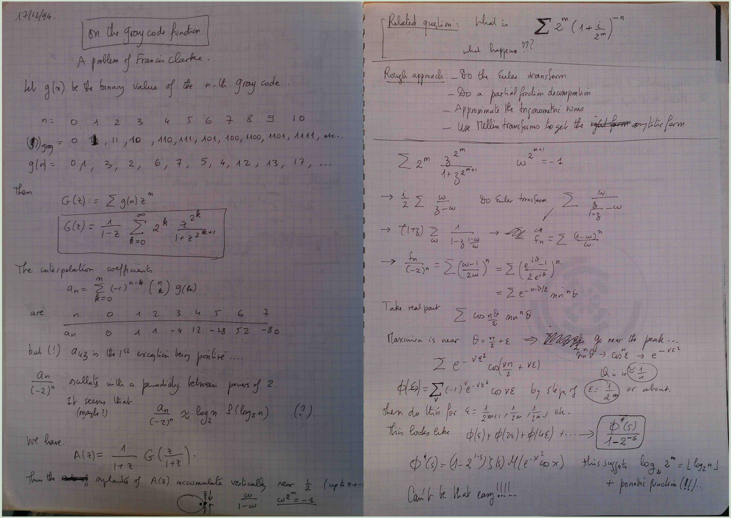
It is just Newton's method applied to P(z) with a close enough starting value, increasing the number of terms in the truncation of P(z) as we proceed. The call to b(7) gives 1357 exact digits in 4 minutes of CPU time.

```
Digits:=16; # must be there because of Maple's idiosyncrasy
b:=proc(m) # computes more than 10*2^m digits of Backhouse's Constant
        local ord, x, i, j, P, DP, t;
        option remember;
        Ithprime:=proc(n) option remember; ithprime(n); end;
        Digits:=15;
        x:=-1/1.45607494858268967;
        ord:=72;
        for i from 1 to m do
                Digits:=2*Digits; ord:=2*ord;
                P:=1; DP:=0; xj:=x;
                for j from 1 to ord do
                        t:=Ithprime(j)*xj; P:=P+t; DP:=DP+t*j; xj:=xj*x;
                od;
                x:=x-P*x/DP;
        od;
 RETURN(-1/x);
 end;
 b(6);  # gives 678 exact digits in 70 seconds
 b(7);  # gives 1357 exact digits in < 4 minutes

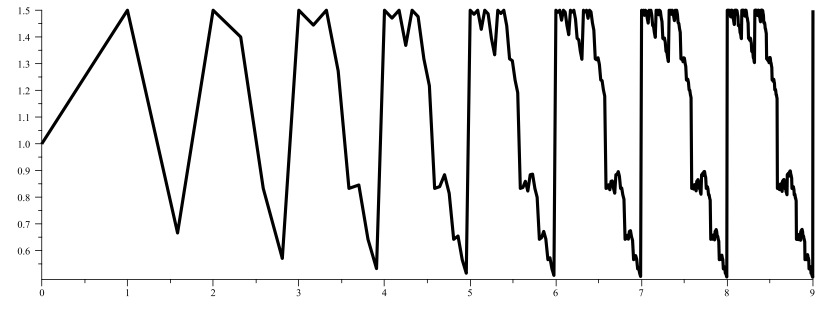
%-----------------------------------------------------------------------
```

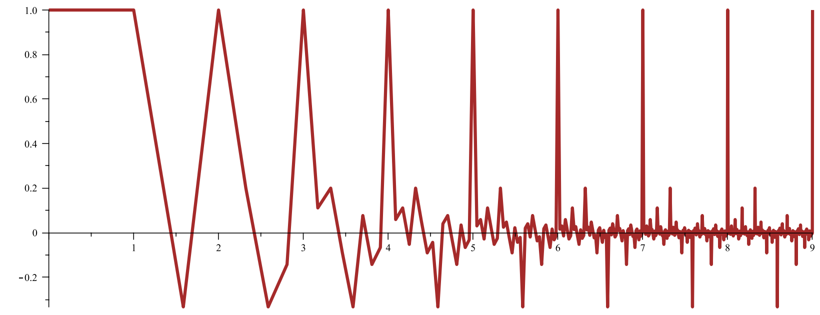
GRAY CODE FUNCTION

$$\sum_{0\leqslant k\leqslant n}\binom{n}{k}(-1)^{n-k}g_k,\; g_n = 2g_{\lfloor \frac{n}{2} \rfloor} + \frac{1-(-1)^{\lceil \frac{n}{2} \rceil}}{2}$$



GRAY CODE FUNCTION: g_n & $g_{n-1} - g_n$



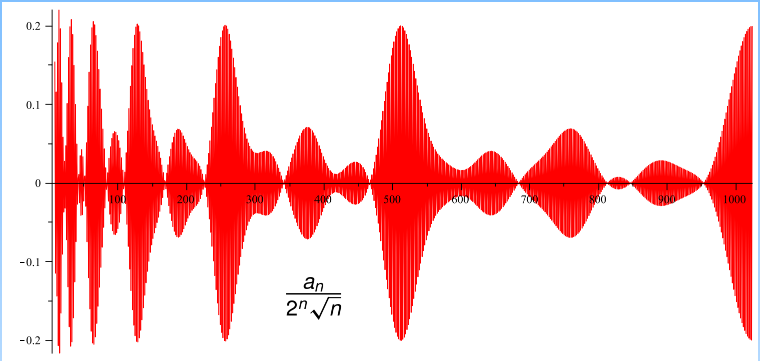


GRAY CODE FUNCTION

$$a_n := \sum_{0\leqslant k\leqslant n} \binom{n}{k} (-1)^{n-k} g(k)$$

$$\frac{a_n}{(-2)^n} = [z^n] G\left(-\frac{z}{2-z}\right) \qquad G(z) := \sum_{j\geqslant 0} \frac{2^j z^{2^j}}{1+z^{j+1}}$$

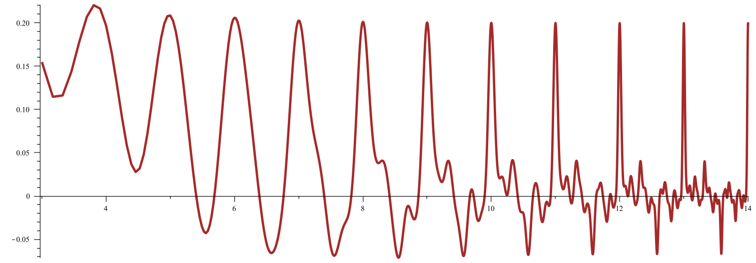
$n = 43$ ***is the first exception that*** $a_n/(-2)^n > 0$



GRAY CODE FUNCTION

$$a_n := \sum_{0\leqslant k\leqslant n} \binom{n}{k} (-1)^{n-k} g(k)$$

$$\frac{a_n}{(-2)^n} = -\frac{1}{2\sqrt{2\pi n}} \int_{-\infty}^{\infty} e^{-(v-n)^2/(2n)} \sum_{3\leqslant k\leqslant L_n+2} \frac{\sin\left(\frac{1}{2} v\pi\right)}{\cos\left(\frac{\pi v}{2^k}\right)}\, dv$$



Asymptotics of $\frac{a_n}{(-2)^n\sqrt{n}}$ remains open

OPEN PROBLEMS IN PF'S OEUVRES

INTRACTABLE

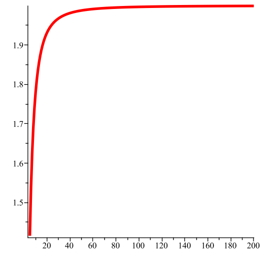
[PF44] (P762) & [PF98] (P217): . . . *ce qui donne lieu à la plus célèbre conjecture de l'informatique*

$$\mathbf{P} \neq \mathbf{NP}$$

[PF197]: $$\left| \sum_{2 \leqslant k \leqslant n} \binom{n}{k} \frac{(-1)^k}{\zeta(k)} \right| = O\left(n^{\frac{1}{2}+\varepsilon}\right)$$

$\equiv$ **Riemann Hypothesis**

RH is also connected to algorithm complexity (with Vallée & Clement): [PF144] [PF157] [PF161]



SOLVED

- [PF51] [PF69] [PF72] [PF108]: *Height & diameter of BSTs* (Devroye, Reed, Drmota, . . .)
- [PF122]: *height of quadtrees* (Devroye)
- [PF112] [PF152]: *Quicksort limit law* (Fill, Janson, Devroye, Neininger, . . .)
- [PF147] *Max deg in planar triangulations* (Gao, Wormald)
- [PF200] $(1-\lambda)^{-\frac{1}{3}}$ *realizable by stochastic context-free grammar?* (Banderier, Drmota)
- . . .

SOLVED?

[PF114] [PF132] [PF144] (with Vallée)
Spectrum of the Euclid transfer operator

Beginning around 1994,
Underlying a series of papers
Stating a set of conjectures
seemingly proven in 2013 by Alkauskas

$$\mathbf{G}_s[f](x) := \sum_{m \geqslant 1} \frac{1}{(m+x)^{2s}} f\left(\frac{1}{m+x}\right)$$

Philippe was interested in computing the spectrum

- **for** $s = 1$**: Euclid algorithm**
- **for** $s = 2$**: Gauss reduction algorithm**

CONJECTURES ON THE SPECTRUM OF $\mathbf{G}_s$

For the Gauss-Kusmin-Wirsing operator $\mathbf{G} := \mathbf{G}_1$

- **All eigenvalues $|\lambda_n|$ are *simple* & *strictly* $\downarrow$**
- **They *alternate* in sign: $(-1)^n\lambda_n > 0$**
- $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = -\phi^2$ **&** $\lambda_n \sim (-1)^{n+1}\phi^{-2n}$

Alkauskas announced in 2013 a proof of the conjectures

arXiv 1210.4083: "***In this work we prove an asymptotic formula for the eigenvalues of L. This settles, in a stronger form, the conjectures of D. Mayer and G. Roepstorff (1988), A. J. MacLeod (1993), Ph. Flajolet and B. Vallée (1995), . . .***"

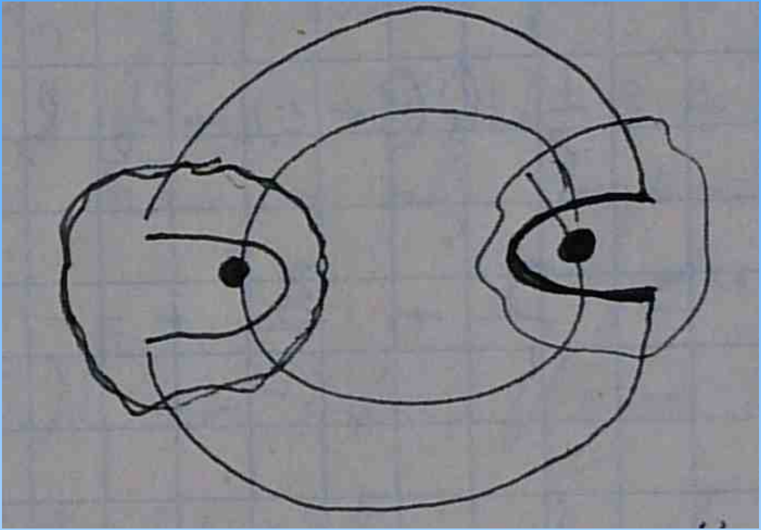
He asked Brigitte if other experiments were performed. Then with Julien, more computations made by Philippe were found . . .

STILL OPEN?

- **[PF207]: non-holonomicity of** $\cos\sqrt{n}, \cosh\sqrt{n}$
- **[PF204]:** ***three-sided prudent polygons,*** $g_4 = 1 + \log_2 3$**?**
- **[PF200]: Buffon machine for Euler's** γ**?**
- **[PF191]:** ***hidden word statistics, convergence rate to normality?***
- **[pF185]: Graeffe polynomials computable at a lower cost?**
- **[PF174]:** ***motif statistics under more general models?***
- **[PF172]: robustness of interconnection in random graphs, finer properties like variance?**

STILL OPEN?

- **[PF161]: trie statistics under general dynamic sources**
- **[PF157]:** ***continued fractions & comparison algorithms, many questions***
- **[PF144]: continued fraction algorithms, uniformity of quasi-power for MGF?**
- **[PF69]:** ***linear worst-case time for tree-matching algorithms?***
- **[PF64]: ambiguity of context free languages, many questions**
- **[PF54]:** ***collision resolution algorithms in random access systems, limit of stability?***



WHAT TO DO WITH THE CAHIERS?

Cahiers ➥ Digital Forms

➥ Web Accessible?

➥ PolyPF Project?