

# LES CAHIERS DE PHILIPPE FLAJOLET

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(with Brigitte Vallée, Julien Clément, ...)

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**Cigarettes**



**Theory**



# OUTSIDE LOOK: 65 NOTEBOOKS



# VERY STRONG “FLAVORS”

✓ *Cigarette smoke*

✓ *Pen ink*



# CONTENTS

*All academic: evolution of ideas, devt of techniques*

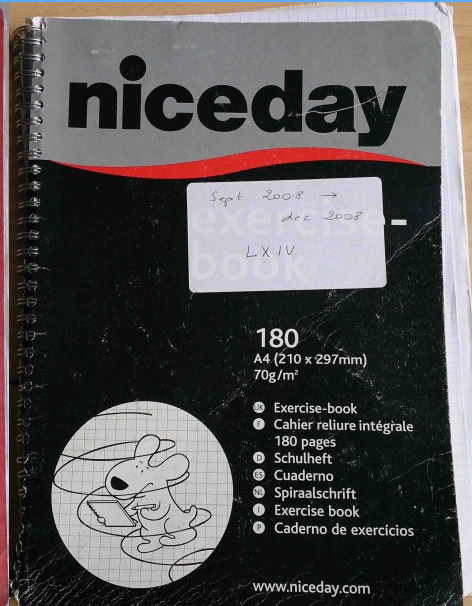
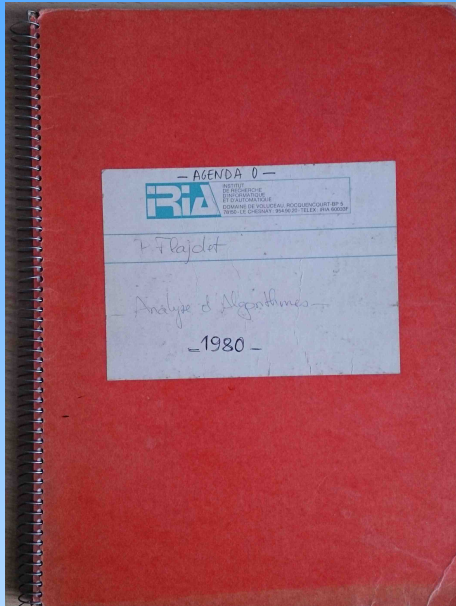
- **Notes/Summaries for talks, lectures, courses, ...**
- **Drafts for papers, book chapters, preprints, ...**
- **Work summary**
- **Maple calculations (symbolic, numerical, figures, tables, expansions)**
- **Email/letter correspondences**
- **Miscellaneous**

## Three categories

- **Finished, well-explored techniques/topics**
- **Unfinished**
- **We-don't-know-yet**



# AGENDA 0 & CAHIER LXIV



# SNAPSHOTS

23 Mars 1980

Les nombres de Stirling de seconde espèce : séries ordinaires et exponentielle

$S_{n,k}$  désigne le nombre de partitions de  $[n]$  en  $k$  classes (blocs) ;

$k! S_{n,k}$  représente ainsi le nombre de surjections de  $[n]$  sur  $[k]$ .

Les deux expressions de séries génératrices (ordinaires ou exponentielles) des  $S_{n,k}$  correspondent à deux présentations différentes des partitions d'ensembles.

$$1) \sum S_{n,k} \frac{z^n}{n!} = \frac{(e^z - 1)^k}{k!}$$

$$2) \sum S_{n,k} z^n = \frac{z^k}{(1-z)(1-2z)\dots(1-kz)}$$

Une preuve directe de ces deux séries génératrices correspond à une preuve combinatoire que 2) en la transformée de Laplace - Borel de 1.

Pour 1), la preuve résulte des manipulations classiques de séries génératrices associées à des mots (cf Analyse d'Algorithmes 1980)

# SNAPSHOTS

30 Dec 83

31 Dec 83 for fun!

## LE POINT DES ARTICLES

Acceptés et revus

2. XPF	Enthalde (espans compis)	accepté	Arche Def	Epr. Dec 83
3. XPF, St	Apas (?) A refer	accepté	BIT	venni Août 84
3. XPF, St	Plaid (?) (lettre ex. d. 760, 1974)	accepté	Inf. d. Antioch	parue

Acceptés

1. R, Sabes	Exposition (rennais en cours d'édit.)	accepté	J. of Alg	à venir
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Soumis

5. R. Parich	Mult. Mim.	à venir	J of Acti	?
6. R. Sl.	Crypt. Calc	non révisé	Mult. Syst. Th.	?
7. R. Re So	Tris: adgeb. meth.	terminé	Annals Dis. Math.	Epr. Août 84
8. R. Fayd J.	Partitels	terminé	IEEE Trans. Th.	accepté à venir
9. R. Of Wo	Bubble mem. (terminé)	terminé	RAIRO	accepté, venu
10. R. Lu Vu	Distribution Harlow	à venir	Math. J. Th. Sc.	?
11. R. Pm3	Rapides	accepté	SIAM Alg. Discrete	?

Terminés, à soumettre

12. R. Fayd Mf	Robuste (à venir)	à venir	Zeitschrift ...?	normis?
13. R. Od	hmbig delat. icatels	terminé	SIAM J. Comp.	terminé

A terminer / réviser

14. R. Sedg	Original Trees	terminé	JARN?	à soumettre fin août 84
15. R. Granberg Ludw	Robuste	terminé	JARN?	à soumettre fin août 84
16. R. Odlyzh	Half of final graphs	à venir	à venir	à venir
17. R. Steyaert	Polyp. Factorization	à venir	à venir	à venir
18. R. Requin	Mellin	à venir	à venir	à venir

En cours de développement

19. R. Pelt	2 Stocks / Pelt + cons.	à venir	à venir	à venir
20. R. Pelt	Quand l'écou. - 2000 admet	à venir	à venir	à venir
21. R. Pelt	21. Pelt (cont.)	à venir	à venir	à venir
22. R. Pelt	22. Pelt (cont.)	à venir	à venir	à venir

## Some Unix tricks & features

(a) echo 'who|wc -l' users logged in

(b) for i in 1 2 3 4  
do  
date >> tmp  
done

(c) for data in "0.1 2.0" "0.2 1.5" \  
"0.3 1.25" do

(date, echo \$data | parval prog) >> results  
echo \$data;  
done

(g) cat > tmp  
1 2 3 bla  
bla  
^D

(h) echo 'expr \$a + 1'  
\$a = 12  
user nouveau nom de variable  
change fois que toto n'afficé

{ toto  
• a = 'cat #rank'  
prog > res. expr \$a + 1'  
} echo 'expr \$a + 1' > #rank

(l) the only rank contient une  
une info, l'incrémenter.

(m) cat ....

(n) tee

(i) if test -f \$1  
then  
echo Yeah! \$1 is there  
fi



# SNAPSHOTS

Predictions (en rouge = valeurs exacts)

par	Rhosh	$\epsilon$	$\gamma$	$\delta$	$\delta$	50	500	5000	50000	500000
[S, *]	40	$10^{-6}$	2.17500	-1.9052		13.47	46.72	151.85	484.44	1536.05
	80	$10^{-6}$	2.1827300	-1.9743		13.45	46.83	152.36	486.09	1541.44
	160	$10^{-6}$	2.17902	-1.9275		13.48	46.79	152.15	485.31	1538.87
	320	$10^{-6}$	2.1843667	-2.02300		13.42	46.82	152.43	486.41	1542.55
	640	$10^{-6}$	2.18011	-1.9128		13.50	46.88	152.23	485.55	1539.58
	1280	$10^{-7}$	2.1843661	-2.02301		13.47957	46.8055	152.23	485.45	1539.20
	1000	$10^{-6}$	2.17941	-1.8732		13.53	46.85	152.23	485.45	1539.20
[* * S]	40	$10^{-6}$	2.50216	-2.5527		31.40	155.07	729.08	3395.41	15760.18
	80	$10^{-6}$	2.44376	-1.468		31.69	152.47	713.09	3315.23	15393.30
	160	$10^{-6}$	2.48516	-2.688		31.04	153.86	723.97	3370.19	15652.82
	320	$10^{-6}$	2.50273	-3.510		30.45	154.15	728.29	3393.22	15762.75
						31.43	155.32			
[SS *]	50	$10^{-6}$				8.65	21.90	50.47	112.01	244.59
	100	$10^{-6}$	3.21582	-3.29		8.55	22.23	51.69	115.18	251.94
	200	$10^{-6}$	3.11956	-2.72		8.76	22.03	50.61	112.19	244.87
						8.6503	22.0128			





# SNAPSHOTS

> bernoulli(# Finds the eigenfunction of degree d for G-uniform

```
f:=proc(d) option remember; local p,i,a;
p:=x^d+sum(a[i]*x^i,i=0..d-1);
expand(p-2^(d-1)*(subs(x=x/2,p)+subs(x=(x+1/2),p)));
{seq(coeff(*,x),i=0..d-1)};
solve(*,{seq(a[i],i=0..d-1)});
subs(*,p);
end;

seq(bernoulli(j),j=0..10);
seq(f(j),j=0..10);
```

```
f :=
proc(d)
local p,i,a;
options remember;
p := x^d+sum(a[i]*x^i,i = 0 .. d-1);
expand(p-2^(d-1)*(subs(x = 1/2*x,p)+subs(x = 1/2*x+1/2,p)
));
{seq(coeff(*,x,i) = 0,i = 0 .. d-1)};
solve(*,{seq(a[i],i = 0 .. d-1)});
subs(*,p);
end
> seq(f(j),j=0..10);
```

$$1, x - \frac{1}{2}x^2 + x^3 - \frac{1}{2}x^4 + x^5 - \frac{3}{2}x^6 - \frac{1}{30}x^7 + x^8 - 2x^9 - \frac{1}{6}x^{10} + \frac{5}{3}x^{11} + x^{12} - \frac{5}{2}x^{13},$$

$$\frac{1}{42}x - \frac{1}{2}x^2 + \frac{5}{2}x^3 + x^4 - 3x^5 - \frac{1}{6}x^6 + \frac{7}{6}x^7 + \frac{7}{2}x^8 + x^9 - \frac{7}{2}x^{10},$$

$$-\frac{1}{30}x - \frac{2}{3}x^2 - \frac{7}{3}x^3 + \frac{14}{3}x^4 + x^5 - 4x^6 - \frac{3}{10}x^7 + 2x^8 - \frac{21}{5}x^9 + 6x^{10} + x^{11} - \frac{9}{2}x^{12},$$

$$\frac{5}{66}x - \frac{3}{2}x^2 + 5x^3 - 7x^4 + \frac{15}{2}x^5 + x^{10} - 5x^{11}$$

> seq(bernoulli(j),j=0..10);

$$1, x - \frac{1}{2}x^2 + x^3 - \frac{1}{2}x^4 + x^5 - \frac{3}{2}x^6 - \frac{1}{30}x^7 + x^8 - 2x^9 - \frac{1}{6}x^{10} + \frac{5}{3}x^{11} + x^{12} - \frac{5}{2}x^{13},$$

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> bernoulli(11,x);

1

les polynômes de Bernoulli sont

ils forment une base des polynômes

9/10/96

$g(f)(x) = \frac{1}{x} \left[ f(x) + f\left(\frac{x+1}{2}\right) \right]$

> bernoulli(# Finds the eigenfunction of degree d for G-uniform

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f:=proc(d) option remember; local p,i,a;
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{seq(coeff(*,x),i=0..d-1)};
solve(*,{seq(a[i],i=0..d-1)});
subs(*,p);
end;

seq(bernoulli(j),j=0..10);
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options remember;
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expand(p-2^(d-1)*(subs(x = 1/2*x,p)+subs(x = 1/2*x+1/2,p)
));
{seq(coeff(*,x,i) = 0,i = 0 .. d-1)};
solve(*,{seq(a[i],i = 0 .. d-1)});
subs(*,p);
end
> seq(f(j),j=0..10);
```

$$1, x - \frac{1}{2}x^2 + x^3 - \frac{1}{2}x^4 + x^5 - \frac{3}{2}x^6 - \frac{1}{30}x^7 + x^8 - 2x^9 - \frac{1}{6}x^{10} + \frac{5}{3}x^{11} + x^{12} - \frac{5}{2}x^{13},$$

$$\frac{1}{42}x - \frac{1}{2}x^2 + \frac{5}{2}x^3 + x^4 - 3x^5 - \frac{1}{6}x^6 + \frac{7}{6}x^7 + \frac{7}{2}x^8 + x^9 - \frac{7}{2}x^{10},$$

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> seq(bernoulli(j),j=0..10);

$$1, x - \frac{1}{2}x^2 + x^3 - \frac{1}{2}x^4 + x^5 - \frac{3}{2}x^6 - \frac{1}{30}x^7 + x^8 - 2x^9 - \frac{1}{6}x^{10} + \frac{5}{3}x^{11} + x^{12} - \frac{5}{2}x^{13},$$

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$$-\frac{1}{30}x - \frac{2}{3}x^2 - \frac{7}{3}x^3 + \frac{14}{3}x^4 + x^5 - 4x^6 - \frac{3}{10}x^7 + 2x^8 - \frac{21}{5}x^9 + 6x^{10} + x^{11} - \frac{9}{2}x^{12},$$

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> bernoulli(11,x);

1



# SNAPSHOTS

From: flajole@nrc.ca  
 Received: by mailz.inria.fr; Fri, 20 Oct 93 20:29:02 +0100  
 Date: Fri, 22 Oct 93 20:29:02 +0100  
 From: Philippe Flajole <flajole@nrc.ca>  
 Message-Id: <9310221929.AA2391@nrc.ca>  
 To: Rejean Valien@nrc.ca, Hervé Guoduniv@can.ca  
 Subject: La realite des choses  
 Cc: flajole@nrc.ca  
 Status: R

celle assez bien a l'analyse. Je fais environ 1.000.000  
 de simulations a la minute de l'algorithme de Gauss.  
 Voici un tableau comparatif entre les probabilites estimees d'un  
 precedent message et les probabilites observees sur un tirage d'un  
 million de mots pourrais faire 100 millions les doigts dans le nez, et  
 gagner un digit de precision supplementaire.  
 La conclusion irreprehensible est donc en tout cas que "mon  
 generateur aleatoire est bon!".

NB: Il s'agit des probabilites cumulees, Pr(D<=k).

k =	Probabilites observees	Predictions
...:1 :	1.000000	1.000000
...:2 :	0.290314	0.289648
...:3 :	0.048423	0.048480
...:4 :	0.010257	0.010278
...:5 :	0.002030	0.002093
...:6 :	0.000423	0.000402
...:7 :	0.000079	0.000079
...:8 :	0.000013	0.000015
...:9 :	0.000001	0.000019
...:10 :	0.000001	0.000037
		0.000051
		0.000069
		0.000087
		0.000105
		0.000123
		0.000141
		0.000159
		0.000177
		0.000195
		0.000213
		0.000231
		0.000249
		0.000267
		0.000285
		0.000303
		0.000321
		0.000339
		0.000357
		0.000375
		0.000393
		0.000411
		0.000429
		0.000447
		0.000465
		0.000483
		0.000501
		0.000519
		0.000537
		0.000555
		0.000573
		0.000591
		0.000609
		0.000627
		0.000645
		0.000663
		0.000681
		0.000699
		0.000717
		0.000735
		0.000753
		0.000771
		0.000789
		0.000807
		0.000825
		0.000843
		0.000861
		0.000879
		0.000897
		0.000915
		0.000933
		0.000951
		0.000969
		0.000987
		0.001005
		0.001023
		0.001041
		0.001059
		0.001077
		0.001095
		0.001113
		0.001131
		0.001149
		0.001167
		0.001185
		0.001203
		0.001221
		0.001239
		0.001257
		0.001275
		0.001293
		0.001311
		0.001329
		0.001347
		0.001365
		0.001383
		0.001401
		0.001419
		0.001437
		0.001455
		0.001473
		0.001491
		0.001509
		0.001527
		0.001545
		0.001563
		0.001581
		0.001599
		0.001617
		0.001635
		0.001653
		0.001671
		0.001689
		0.001707
		0.001725
		0.001743
		0.001761
		0.001779
		0.001797
		0.001815
		0.001833
		0.001851
		0.001869
		0.001887
		0.001905
		0.001923
		0.001941
		0.001959
		0.001977
		0.001995
		0.002013
		0.002031
		0.002049
		0.002067
		0.002085
		0.002103
		0.002121
		0.002139
		0.002157
		0.002175
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		0.002211
		0.002229
		0.002247
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		0.002661
		0.002679
		0.002697
		0.002715
		0.002733
		0.002751
		0.002769
		0.002787
		0.002805
		0.002823
		0.002841
		0.002859
		0.002877
		0.002895
		0.002913
		0.002931
		0.002949
		0.002967
		0.002985
		0.003003
		0.003021
		0.003039
		0.003057
		0.003075
		0.003093
		0.003111
		0.003129
		0.003147
		0.003165
		0.003183
		0.003201
		0.003219
		0.003237
		0.003255
		0.003273
		0.003291
		0.003309
		0.003327
		0.003345
		0.003363
		0.003381
		0.003399
		0.003417
		0.003435
		0.003453
		0.003471
		0.003489
		0.003507
		0.003525
		0.003543
		0.003561
		0.003579
		0.003597
		0.003615
		0.003633
		0.003651
		0.003669
		0.003687
		0.003705
		0.003723
		0.003741
		0.003759
		0.003777
		0.003795
		0.003813
		0.003831
		0.003849
		0.003867
		0.003885
		0.003903
		0.003921
		0.003939
		0.003957
		0.003975
		0.003993
		0.004011
		0.004029
		0.004047
		0.004065
		0.004083
		0.004101
		0.004119
		0.004137
		0.004155
		0.004173
		0.004191
		0.004209
		0.004227
		0.004245
		0.004263
		0.004281
		0.004299
		0.004317
		0.004335
		0.004353
		0.004371
		0.004389
		0.004407
		0.004425
		0.004443
		0.004461
		0.004479
		0.004497
		0.004515
		0.004533
		0.004551
		0.004569
		0.004587
		0.004605
		0.004623
		0.004641
		0.004659
		0.004677
		0.004695
		0.004713
		0.004731
		0.004749
		0.004767
		0.004785
		0.004803
		0.004821
		0.004839
		0.004857
		0.004875
		0.004893
		0.004911
		0.004929
		0.004947
		0.004965
		0.004983
		0.005001
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		0.005073
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		0.005109
		0.005127
		0.005145
		0.005163
		0.005181
		0.005199
		0.005217
		0.005235
		0.005253
		0.005271
		0.005289
		0.005307
		0.005325
		0.005343
		0.005361
		0.005379
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		0.005415
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		0.005577
		0.005595
		0.005613
		0.005631
		0.005649
		0.005667
		0.005685
		0.005703
		0.005721
		0.005739
		0.005757
		0.005775
		0.005793
		0.005811
		0.005829
		0.005847
		0.005865
		0.005883
		0.005901
		0.005919
		0.005937
		0.005955
		0.005973
		0.005991
		0.006009
		0.006027
		0.006045
		0.006063
		0.006081
		0.006099
		0.006117
		0.006135
		0.006153
		0.006171
		0.006189
		0.006207
		0.006225
		0.006243
		0.006261
		0.006279
		0.006297
		0.006315
		0.006333
		0.006351
		0.006369
		0.006387
		0.006405
		0.006423
		0.006441
		0.006459
		0.006477
		0.006495
		0.006513
		0.006531
		0.006549
		0.006567
		0.006585
		0.006603
		0.006621



# SNAPSHOTS

## Discrete versus Continuous Mathematics: Transgressing the Boundaries

Philippe FLAJOLET  
December 23, 2008

Recent decades have seen a surge of interest in discrete mathematics and combinatorics, where what is at stake is the study of properties of finite objects constructed by a finite set of rules. Structural combinatorics is, now, visible part of the iceberg in Academics, with markedly profound results, such as the theory of minors or the long-sought solution to the perfect graph conjecture. The problem there is to answer questions such as: Given such and such construction rules, which properties must the corresponding objects invariably satisfy?

Partly pushed by the needs of several branches of science—from computer science to probability theory to bio-informatics to statistical physics—the past twenty to thirty years have seen the appearance of a large number of studies dedicated to another (at least) equally fundamental question: Given such and such construction rules, which properties must the corresponding objects satisfy in an overwhelming proportion of cases? In other words, rather than seeking what is always true, we want to characterize what is almost always true. In this discrete combinatorial world, we want to measure things.

My own research since the early 1990's has been precisely a sustained effort meant to address the characterization of what is "almost always" true amongst the most important structures of mathematics, such as words, trees, mappings, graphs, maps, and permutations. It has constantly alternated between methodological works (e.g., the development of singularity analysis with Odlyzko, the elucidation of the power of the Mellin transform) and research dedicated to solving concrete problems arising from applications (the analysis of sequences in relation to probabilistic counting algorithms). An extreme is the publication by Cambridge University Press (at the turn of 2008-2009) of an 825 pages monograph co-authored with Sedgewick and titled *Analytic Combinatorics*, which I will abbreviate as *AC*.

The field of analytic combinatorics as expounded in the book *AC* constitutes the basic layer on which the present proposal is built. Roughly, the major theme is that a class of combinatorial structures is reduced to a locally smooth surface (the Riemann surface of a corresponding generating function), whose "cracks" (the signified name is singularities) are seen to contain a host of quantitative information. For instances, as I showed with Odlyzko in 1982 and 1994, the complex-analytic structure of the iteration  $z_{n+1} = z + z^n$  near

<sup>1</sup>In December 2008, right before the release of *AC* (quipped by some as "an best-order requirement article"), the Google Scholar citation score of the manuscript (which had been on the web for several years) was 240, including self-citations: this appears to be substantially more than the average citation record an existing mathematics book.



## Merry Christmas

**THEOREM.** Define the Hurwitz numbers [Math. Ann. 51 (1899), 196-206] by

$$E_n = \frac{(4n)!}{1^{4n}} \sum_{\nu \in \mathbb{Z}^n} \frac{1}{(\nu + \mathbf{N}(-1))^{4\nu}}, \quad \Omega := \frac{1}{\sqrt{2\pi}} \Gamma\left(\frac{1}{2}\right).$$

The ordinary generating function of  $(E_n)$  satisfies

$$10 \sum_{n \geq 1} E_n z^{n-1} = \frac{1}{1 - \frac{\Omega \cdot z}{1 - \Omega \cdot z}}$$

$$c_n = \begin{cases} \frac{1}{10} \frac{(2n)(2n+1)!(2n+2)!}{(4n+3)(4n+5)(4n+1)(4n+3)} & (n \text{ even}) \\ \frac{1}{10} \frac{(4n+1)!(4n+3)!(4n+5)!}{(2n+1)!(2n+2)!(2n+3)!} & (n \text{ odd}) \end{cases}$$

where  $c_1 = -3$ ,  $c_2 = \frac{16}{15}$ , and  $c_n = 0$  for  $n \geq 3$ .



# SNAPSHOTS

27 DEC 2008

$D_n$  is complex of WLF

(De Wammeker et al, JCTA, 1997)

Define  $D_n = \sum_{k=0}^n (-1)^k \binom{n}{k}$  with EEF  $\exp(1 - e^x)$ .

Conjecture  $D_n \neq 0$  for all  $n > 2$ .

[DeLaO:07] approach the problem by  $p$ -adic analysis. They<sup>\*</sup> provide exact for  $n \equiv \alpha_0 \pmod{4}$  or  $n \equiv \alpha_1 \pmod{4}$  where  $\alpha_0 = 2$ ,  $\alpha_1 = 2 \times 3 \times 131 \times 593$  and  $H = 3 \cdot 2^{20}$ .

• Can we proceed by continued fractions (and associated OPS + congruences)?

\* these guys have a good table-topical list - known jobs.

$$\sum_{k=0}^n (-1)^k \binom{n}{k} z^k = \frac{1}{1 - uz - \frac{u^2 z^2}{1 - (u+1)z - \frac{2uz^2}{\vdots}}}$$

$$(n=1) \Rightarrow \frac{1}{1+z + \frac{1+z^2}{1+2z + \frac{2z^2}{1+(-1)z + \frac{3z^2}{\vdots}}}}$$

admits Stieltjes form. Aah!

Alts  
 (1)  $n$  variants of Poisson-Charlier

$$\sum_{k=0}^n D_n z^k \quad \text{don't have conv coeffs?}$$

$$n \equiv 2 \pmod{4}$$

↳ for standard Bell #'s you

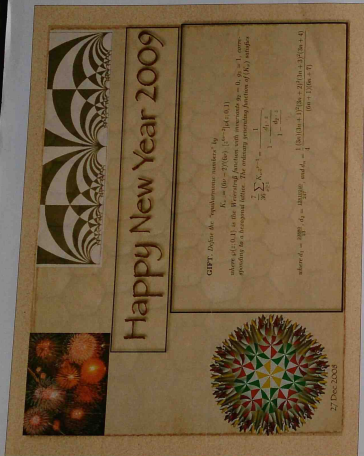
$$1, -1, 1, -1, \frac{1}{2}, -2, \frac{1}{6}, -6, \frac{1}{24}, -24, \frac{1}{120}, -120, \frac{1}{720}, -720, \dots$$

NB the unnormalized continued fraction (Aplg) is

$$\frac{1}{1 + \frac{z}{1 + \frac{z}{-1 + \frac{z}{-1 + \frac{z}{2 + \frac{z}{2 + \frac{z}{-1 + \frac{z}{-6 + \frac{z}{-6 + \frac{z}{\vdots}}}}}}}}}}$$

where coeffs are alternatingly

$$\frac{(-1)^k}{n!} \quad (-1)^k \cdot n!$$



# INSIDE LOOK: AGENDA I

- *Stirling #s of the 2nd kind*
- Symbolic method in analysis of tree algorithms
- *Remarks on partitions*
- Trie statistics
- *q-Laguerre polynomials*
- *Simon-Newcomb problem*
- Lexicographic tree height
- Search tree height
- Approximate counting
- Complexity calculus
- Markov chains
- Exp-variate generation
- Talk by Guy Fayolle
- Extension of approximate counting
- Asymptotics/Mellin
- *Distribution of path areas*
- Pippenger's communication protocols
- *Combinatorial sums asymptotics*
- DST asymptotics
- Grid file algorithms
- *Functional graphs*
- *Differential equations & linear systems*



# AGENDA I

1982/6/6: after visit to Bell Labs, initiated random mapping statistics with Odlyzko

## RANDOM MAPPING STATISTICS

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**Abstract.** Random mappings from a finite set into itself are either a heuristic or an exact model for a variety of applications in random number generation, computational number theory, cryptography, and the analysis of algorithms at large. This paper introduces a general framework in which the analysis of about twenty characteristic parameters of random mappings is carried out: These parameters are studied systematically through the use of generating functions and singularity analysis. In particular, an open problem of Knuth is solved, namely that of finding the expected diameter of a random mapping. The same approach is applicable to a larger class of discrete combinatorial models and possibilities of automated analysis using symbolic manipulation systems ("computer algebra") are also briefly discussed.

*Initiated in 1982  $\implies$  Published in 1990*



# DEEPER LOOK

**Flajolet, Knuth & Pittel (1989)  
The first cycles in an evolving graph**



**Backhouse's constant**

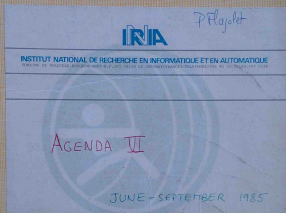


**Gray code function**





# FIRST CYCLE IN EVOLVING GRAPHS



HERAKLES

180 PAGES

August 15th, 1985

## WAITING FOR CYCLES

CYCLES IN A RANDOM GRAPH EVOLUTION MODEL

The following problem was mentioned as open (and due to Erdős) at the Random Graph conference in Poitiers (Aug 1985, July Ed. Palmer) maybe Richard Merson-Davies

Take a set of  $m$  nodes (disconnected, i.e.  $K_0$ ) and throw in edges at random until a circuit occurs. How large is the circuit?

First appeal: Determining stopping times...

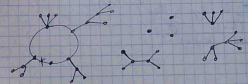
Notations:  $m = \#$  of points  $N = \binom{m}{2}$  (there are no self-loops)

Points are assumed to be labelled from 1 to  $m$  by distinct integers

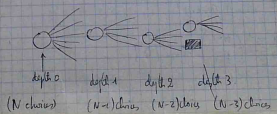
A configuration (abundant, reduced) of size  $m$  is formed of:

- a cycle of trees with one edge on the cycle marked
- a set of disconnected trees (node disjoint)

if cycle has length  $\geq 3$ , trees are not rooted, cycle is not oriented



The process: at each stage, each of the possible edges is equally likely chosen



65



# FIRST CYCLE IN EVOLVING GRAPHS

An extended (edge time-stamped) configuration is a reduced configuration with edges labelled in order 1, 2, 3, ..., the marked edge having the largest label.

- Claims:
- (1) Each terminal node of the process tree is identified by an extended configuration.
  - (2) Each extended configuration at depth  $k$  has probability  $\frac{1}{N(N-1) \cdot (N-k+1)}$ .
  - (3) To each standard configuration, there correspond exactly  $(k-1)!$  extended configurations, each equally likely.

Thus:

Lemma: If  $C_{n,k}$  is the number of standard configurations on  $n$  nodes with  $k$  edges, then

$$P_{n,k} = \frac{(k-1)!}{N(N-1) \cdot (N-k+1)} C_{n,k}$$

↑  
stepping probability

And we have the other form:

Lemma: Let  $\mathcal{Q}$  be a property of standard configurations,  $P_{n,k}(\mathcal{Q})$  the probn that the process stops with  $\mathcal{Q}$  satisfied,  $C_{n,k}(\mathcal{Q})$  the # of standard configs satisfying  $\mathcal{Q}$ , then:

$$P_{n,k}(\mathcal{Q}) = \frac{(k-1)!}{N(N-1) \cdot (N-k+1)!} C_{n,k}(\mathcal{Q})$$

and

$$P_n(\mathcal{Q}) = \frac{1}{N} \sum_k \binom{N-1}{k-1} C_{n,k}(\mathcal{Q})$$

Verification:  $n=3$   $N=3$



3 standard configs:

$$P_{3,3} = \frac{2!}{3 \cdot 2} \cdot 3 = 1$$

$$C_{3,3} = 3$$

$n=4$   $N=6$



$$\# \text{ configs} = 4 \times 3 = 12$$

$$C_{4,3} = 12$$

↑  
isolated pt.  
↑  
marked edge



$$\# \text{ configs} = 4 \times 3 \times 3 = 36$$

$$C_{4,4} = 48$$

↑  
isolated pt.  
↑  
marked edge  
↑  
bridge pt-to-bridge



$$\# \text{ configs} = \frac{4}{2} \cdot 3! \cdot 4 = 12 \cdot 4 = 48$$

↑  
circulation points with orientation killed  
↑  
marked edge

$$\Rightarrow P_{3,3} = 1 \quad P_{4,3} = \frac{2!}{6 \cdot 5 \cdot 4} \cdot 12 = \frac{1}{5} \quad P_{4,4} = \frac{3!}{6 \cdot 5 \cdot 4 \cdot 3} \cdot 48 = \frac{4}{5}$$

Counting the number of standard configurations (exp. gen. fun.)

$$\text{Let } Y(t) = t e^{Y(t)} \quad Y(t) = \sum_{n \geq 1} \frac{t^n}{n!} \cdot n! \cdot \text{gen. fun. for rooted trees}$$

$$y(t) = \sum_{n \geq 1} \frac{t^n}{n!} \cdot n! \cdot \text{gen. fun. for unrooted trees}$$

then

$$C(3) = \frac{1}{2} \frac{Y'(3)}{1-Y(3)} e^{Y(3)}$$

since  $\frac{1}{2}$  with the cycle orientation  
+ a marked cycle = an open cycle  
= a list of trees.  
+  $e^Y$  once a list of trees.

Now marking edges: observe that in each tree

$$\# \text{ edges} = \# \text{ nodes} - 1$$

# FIRST CYCLE IN EVOLVING GRAPHS

Then

$$\sum_{n=0}^{\infty} C_n u^k z^n = \frac{1}{n!} \frac{Y'(zu)}{1-Y(zu)} e^{\pm Y(zu)}$$

But, as remarked by J.W. Moon  $y(t) = Y(t) - \frac{1}{2} Y'(t)$  then

$$C(u, z) = \frac{1}{2} \frac{Y'(zu)}{1-Y(zu)} e^{\pm(Y(zu) - Y'(zu)/2)}$$

with Ser. (1/2) more term.

which seems to start correctly as

$$\frac{z^3}{3!} 3u^3 + \frac{z^4}{4!} (12u^3 + 48u^4) + O(z^5)!$$

Continuation: Some random ideas.

(A) A variant of Lagrange's formula should be useful:

$$\frac{1}{2\pi i} \int_{\gamma} F(y) \frac{dz}{z^{n+1}} = \frac{1}{2\pi i} \int F(y) (1-y) e^{-y} \frac{dy}{y^{n+1} e^{-(n+1)y}}$$

$$\text{let } z = Ye^{-y} \quad dz = (1-y)e^{-y} dy$$

$$[z^n] F(y) = [Y^n] F(y) (1-y) e^{-ny}$$

(B) How to do the probability weighting?

let  $A(z, u) = \sum_{n,k} a_{n,k} u^k z^n$ . We want

$$B(z) = \sum_n \left( \sum_k \binom{n}{k} u^k \right) z^n \quad N = N(u) = \binom{n}{u}$$

let  $U(z) = \sum (1-u) \binom{n}{u} z^n$ . Then

$$A(z, u) \odot U(z) = \sum a_{n,k} u^k (1-u) \binom{n}{u} z^n \quad (\text{Hadamard prod})$$

and integrate from 0 to 1 using (maybe  $\sum_{n,k} \binom{n}{u} u^k (1-u) \binom{n}{u} z^n \dots$ )

$$\int_0^1 u^{\alpha-1} (1-u)^{\beta-1} du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \frac{\Gamma(\alpha-1)\Gamma(\beta-1)}{(\alpha+\beta-1)!}$$

That does it!!! Next, maybe we technique in G. Kreweras' paper.

looks promising!!! To be done!!!

The exact form of the probability distribution

$$[t^n] \frac{Y(t)}{1-Y(t)} e^{\pm(Y(t) - Y'(t)/2)}$$

$$= [Y^n] Y^{\pm 2} e^{\left(\frac{1}{2} + n\right)Y - Y'/2} = [Y^{n-3}] e^{\left(\frac{1}{2} + n\right)Y - Y'/2}$$

$$= [Y^{n-3}] \sum_{\ell} \frac{Y^{\ell}}{\ell!} \left( \left(\frac{1}{2} + n\right) - \frac{Y}{2u} \right)^{\ell}$$

$$= [Y^{n-3}] \sum_{\ell, j} \frac{Y^{\ell}}{\ell!} \left(\frac{1}{2} + n\right)^{\ell-j} \left(\frac{1}{2u}\right)^j Y^j$$

$$= \sum_{2\ell+j=n-3} \frac{1}{\ell! j!} \left(\frac{1}{2}\right)^{\ell-j} \left(\frac{1}{2u}\right)^j$$

$$\Rightarrow C(z, u) = \frac{1}{2} \sum_n z^n u^n \sum_{\substack{\ell, j \\ 2\ell+j=n-3}} \frac{1}{\ell! j!} \left(\frac{1}{2}\right)^{\ell-j} \left(\frac{1}{2u}\right)^j$$

$$[z^n] C(z, u) = \frac{n!}{2} u^n \sum_{2\ell+j=n-3} \frac{1}{\ell! j!} \left(\frac{1}{2}\right)^{\ell-j} u^{j-\ell} u^{-n} j^{-r}$$

$$[u^n] \dots = [u^n] \dots$$

with 2 as independent parameter

$$\begin{cases} j = 2\ell - n + 3 \\ 2\ell - j = n - 3 \\ \ell - j = n - 3 - \ell \\ k = n - (\ell - j) - r \\ r = n - (\ell - j) - k \\ \ell - r = \ell + k - n \end{cases}$$

$$C_{n,k} = \sum_{j, \ell, r} \frac{n!}{2} \frac{(-1)^{\ell-j}}{(\ell-j)! (j-r)!} \left(\frac{1}{2}\right)^{\ell-j} n^{j-r}$$

$$C_{n,k} = \frac{n!}{2} \sum_{\ell} \frac{(-1)^{\ell-j}}{(2\ell-j)! (\ell+k)!} \left(\frac{1}{2}\right)^{\ell-j} n^{2\ell-k-\ell}$$

$$C_{3,5} = 3$$

$$C_{4,3} = 12$$

$$C_{4,1} = 68$$

$$\text{index is } \max\{k-3, u-k\} \leq \ell \leq n-3$$



# FIRST CYCLE IN EVOLVING GRAPHS

17/1/85

Summary The main steps

Assume we wish to evaluate the probability  $p_n$  of an event  $\mathcal{A}$ ; for  $n=0$  or  $n=1$  we can do this by hand,  $c$  is a finite integer.

- 1- First set up purely combinatorial equations for the total of configurations  $\mathcal{Q}$  and then an equation for  $p_n$  with nodes  $\mathcal{Z}$  and edges summed by  $\mathcal{Q}$ . If  $\mathcal{Q}_n, p_n \neq \#$  - then go through of eqs for  $p_n$ .

$\mathcal{Q}(z, u) = \sum \mathcal{Q}_n u^n z^n$  (1)  
 will usually have a closed form in terms of the comb.  $\mathcal{Y}(z)$  defined implicitly by

$$\mathcal{Y}(z) = z e^{\mathcal{Y}(z)} \quad (2)$$

For a fixed  $u$ , fixed  $\#$  of nodes  $n$  we have

$$\mathcal{Q}_n(u) = n! [z^n] \mathcal{Q}(z, u) \quad (3)$$

- 2- Go from combinatorial counts to process of adding edges. It turns out that the probability  $p_n$  is obtained by a simple calculation of coefficients of  $\mathcal{Q}_n$ :  
 $p_n = \frac{1}{n!} \mathcal{Q}_n(u)$

show the  $h_n$  equation has an integral representation

$$\mathcal{Q}_n(u) = \int_0^1 \dots \quad (4)$$

- 3- For parameters of interest, evaluate the integral above by Laplace method for integrals. Because of the localizing behavior  $(1-u)^{2n}$  in the above field, integrals should be expanded to some nearby base value of  $u = 0(\frac{1}{n})$ . This process will not stop.

- 4- Evaluate  $\mathcal{Q}_n(u)$  when  $u$  is  $O(\frac{1}{n})$ . To that purpose we can use integral formula

$$\mathcal{Q}_n(u) = \frac{1}{2\pi i} \int \mathcal{Q}(z, u) \frac{dz}{z^{n+1}} \quad (5)$$

and ~~the~~ performance as a function of  $n$  (and  $u$ ) akin to the proof of Laplace's lemma

The integral (5) bears of the form

$$\frac{1}{2\pi i} \int \mathcal{F}_n(u, \gamma) d\gamma \quad (6)$$

and evaluate this integral by the saddle pt method. This can be related to Laplace's lemma on coeff of implicitly defined function (the saddle pt is due to  $\gamma$  integral approach)

- 4- Once the Laplace method has been explained, use then to evaluate (4) using the Laplace method of integrals to evaluate (4).

Generalize how we do this. It seems that the peak for application of S.P. should appear when  $v = \frac{1}{n}$ . So we set

$$v = \frac{1}{n}$$

$$\text{let } I = \int_0^1 A\left(\frac{u}{1-u}\right) (1-u)^{2n} \frac{du}{u} \quad (7)$$

$$I = n! \int_0^1 (1-u)^n \left(\frac{u}{1-u}\right)^n \left\{ \int \mathcal{Q}\left(\frac{u}{1-u}, \gamma\right) d\gamma \right\} \frac{du}{u} \quad (8)$$

$$\text{Set } v = \frac{u}{1-u} \quad dv = \frac{du}{(1-u)^2} \quad u = \frac{v}{1+v} \quad du = \frac{dv}{(1+v)^2} \quad 1-u = \frac{1}{1+v}$$

$$I = n! \int_0^\infty \frac{1}{(1+v)^{n+1}} \frac{v^{n+1}}{(1+v)^{n+1}} \left\{ \int \mathcal{Q}(v, \gamma) d\gamma \right\} \frac{dv}{(1+v)^2}$$

$$= n! \int_0^\infty \frac{v^{n+1}}{(1+v)^{2n+1}} \left\{ \int \mathcal{Q}(v, \gamma) d\gamma \right\} \frac{dv}{v(1+v)}$$

$$\text{Set } v = \frac{\lambda}{n} \quad \text{then } \frac{dv}{v} = \frac{d\lambda}{\lambda}$$

$$I = n! \int_0^\infty \frac{\lambda^{n+1}}{\left(1 + \frac{\lambda}{n}\right)^{2n+1}} \left\{ \int \mathcal{Q}\left(\frac{\lambda}{n}, \gamma\right) d\gamma \right\} \frac{d\lambda}{\lambda \left(1 + \frac{\lambda}{n}\right)} \Rightarrow \text{replace by 2}$$

$$I \sim n! \int_0^\infty e^{n(\log \lambda - \frac{\lambda}{n})} e^{-\frac{\lambda^2}{n}} \left\{ \dots \right\} \frac{d\lambda}{\lambda}$$

Actually  $\int_0^\infty$  is improper only if  $\int$  term vanishes!!!



# FIRST CYCLE IN EVOLVING GRAPHS

Note: Expect  $\lambda \leq 2$  to give most of contribution from

$$(1-u)^N (1-u)^K \text{ is maximised when } u \approx \frac{K}{N} \approx \frac{2}{m}$$

That things should happen when  $\lambda \leq 1$ .

Good hope: saddle point for integral with  $\frac{1}{2} \frac{d^2 e}{dx^2}$  appears when  $\lambda \approx 2$  which ~~is~~ is such that terms in  $e^{...}$  are cancelled by the other ones...!!! Everything looks promising [ASIAN]

Take for instance path that cycle has left  $c$  then plug into (I)

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2} \lambda^{c+1} (1-y)^c e^{\frac{\lambda}{2} (y-y^2) + n \log y} dy$$

but saddle point appears when  $\frac{\lambda}{2} (1-y) - \frac{c}{y} (1-y) = 0 \Rightarrow$  take  $y = \lambda$ .

$$P_c \sim n! n^{-n} \int_0^{\lambda} e^{n(\log t + \frac{\lambda}{2} t^2 - \frac{t}{\lambda})} dt \sim e^{-n(\log \lambda + \frac{\lambda}{2} - \frac{1}{\lambda})} e^{-\frac{n}{2} \lambda^2} e^{\frac{n}{\lambda}}$$

$$P_c \sim \frac{n! n^{-n}}{\sqrt{2\pi n}} \int_0^1 \frac{1}{2} \lambda^{c+1} \sqrt{1-\lambda} e^{\frac{\lambda}{2} (1-\lambda^2)} d\lambda$$

GOSH!!!

Now this  $\sum_{c=0}^{\infty} \lambda^c$  is 2 contributors for  $\lambda > 1$  HAS TO BE negligible!  
- A persistence -

Expected cycle length is tricky because of pole as  $\lambda \rightarrow 1$ . One needs further expansion!!! (for worse denominator  $\frac{1}{\sqrt{2\pi n}}$  vanishes!!!) Problem seems to be

$$\text{Sketch of } \int_0^1 e^{N \lambda^{3/2} / 2} d\lambda \sim \frac{1}{\sqrt{(1+\lambda) + \frac{\lambda^2}{(1-\lambda)^2}}} \times \frac{1}{1-\lambda}$$

If my expectations are wrong (bad cycle!!!)

SADDLE POINTS:

$\lambda < 1$	$Y \approx \lambda$
$\lambda = 1$	$Y \approx 1 - n^{-1/2}$
$\lambda > 1$	$Y \approx 1 - n^{-1/2} (1 - \frac{1}{\lambda})^2$

$$\phi(n) \sim \frac{e^{3/4}}{2} \int_{-1}^1 \frac{d\lambda}{1 - e^{-\sqrt{1+(1-\lambda)^2} n}}$$

$$\sim \frac{e^{3/4}}{2} \int_0^1 \frac{dx}{\sqrt{1+nx^2}} \quad \text{let } nx^2 = u \quad x = \frac{\sqrt{u}}{n} \quad dx = n^{-1/2} \frac{1}{2} u^{-1/2} du$$

$$\sim \frac{e^{3/4}}{2} \int_0^1 \frac{1}{\sqrt{1+nx^2}} dx \sim \frac{1}{\sqrt{2\pi n}} \int_0^{\infty} \frac{du}{\sqrt{1+(1+u)^2}} n^{1/2} \times \frac{1}{\sqrt{2\pi n}}$$

$$E(K_n \sim \sqrt{\frac{2n}{\pi}} e^{3/4} \int_0^{\infty} \frac{du}{\sqrt{1+(1+u)^2}} n^{1/2})$$

WOW!!!  
Gosh...  
Saturday...  
Greece!!!

(differentiate) +  $\frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1)} \rightarrow \sqrt{\pi}$

NB: Slightly wrong in that  $\sum_{c=1}^{\infty} c! n^{-c} \sim \sum_{c=1}^{\infty} \frac{c}{c! n} \sim \sum_{c=1}^{\infty} c^{-2} \sim \frac{1}{2n}$ !!!  
To be checked!!!

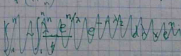
An effective bound for  $\lambda \geq 2$  on  $B_n(u)$ :

$$B_n(u) = \frac{1}{2\pi i} \int_{\gamma} z^2 e^{\frac{1}{2}(4-y^2)u} e^{-ny} \frac{dy}{y^{n+1}}$$

Take  $|y|=1$  as circle of integration then

$$B_n(u) \leq \frac{1}{2} e^{\frac{1}{2} u} e^{-n} \quad (\text{all } u!)$$

$$\text{Then } \int_0^{\infty} \frac{\lambda^n}{(1+\frac{\lambda}{2})^n} B_n(\frac{\lambda}{2}) d\lambda \sim \int_0^{\infty} \frac{1}{2} e^{-\lambda/2} d\lambda \leftarrow \text{small since } \times 2$$



Observation:  $B_n(u)$  increases,  $u^2 B_n(\frac{1}{2})$  is maximal  $c(u)$  is not maximal. One should expect for the peak  $\frac{1}{2} B_n(u)$  the following graph:  $\frac{1}{2} B_n(u)$   $\leftarrow$  large contribution?

Such expansions bubble probably

Always  $B$  uniformly in  $\lambda$  (S)  $\Rightarrow$  want to check  $B$  uniformly in  $\lambda$  (S)



# FIRST CYCLE IN EVOLVING GRAPHS

August 18, 1985

Some remarks on Odlyzko-Tambara Theorem

Aim is to prove:

Theorem: let  $f(z)$  be analytic for  $|z| \leq 1$  with the exception of a single singularity at  $z=1$ . Assume that  $f(z)$  satisfies (A) & (B)

$$f(z) = O\left(\frac{1}{(1-z)^\alpha} L\left(\frac{1}{1-z}\right)\right) \text{ in a neighbourhood of } |z|=1 \text{ with } |z| < 1$$

where

- (i)  $L$  is an increasing function  $\rightarrow \infty$
- (ii)  $L$  is slowly increasing, that is to say

$$\forall c > 0 \quad \frac{L(cx)}{L(x)} \rightarrow 1 \text{ as } x \rightarrow \infty$$

Then

$$[z^n] f(z) = O\left(n^{\alpha-1} L(n)\right)$$

Example are

$$[z^n] O\left(\frac{1}{(1-z)^\alpha} \log \log \frac{1}{1-z}\right) \Rightarrow O(n \log \log n)$$

$$[z^n] O\left(\frac{1}{(1-z)^{1/2}} \sqrt{\log \frac{1}{1-z}}\right) = O(n^{1/2} \sqrt{\log n})$$

Proof: use lemma

$$\exists g(x) : g(x) \rightarrow \infty \text{ s.t. } \frac{L(xg(x))}{L(x)} \rightarrow 1$$

Then use the Riemann method.

$$\int_{r_1}^{r_2} \frac{1}{z} f(z) dz = O\left(\frac{1}{n} \times n^\alpha L(n)\right)$$

$$\int_{r_2}^1 \frac{1}{z} f(z) dz = \int_{r_2}^1 \frac{1}{\theta^n} L(\theta) d\theta = n^{\alpha-1} L(n)$$

(NB) Use only the fact that  $L$  is increasing.

Problem is: What about slowly decreasing function?

Application (A)  $(1-z)^\alpha \log(1-z)$  appears in partial math

(B)  $\frac{1}{\sqrt{1-z} \log \frac{1}{1-z}}$  appears in the comparison

(C)  $\sqrt{1-z} \frac{\log \frac{1}{1-z}}{\log \log \frac{1}{1-z}}$  appears in maximum degree for trees.

Observation: Cycle  $\left\{ \begin{array}{l} \alpha > 1 \\ \alpha \leq 1 \end{array} \right\}$ ,  $L$  slowly  $\left\{ \begin{array}{l} \text{increasing} \\ \text{decreasing} \end{array} \right\}$

Can we get the exact form of coefficients? It should be yes! but then requires to give in the complex plane with additional condition

Return me to mobile calculation:  $n=5$   $N = \binom{5}{3} = 10$

$K=3$



# Pass

Passes

30 pass

$$\frac{30 \text{ pass}}{10 \cdot 3} = \frac{2}{10} \times 30 = \frac{1}{2}$$

$$\approx 0.0533$$

$K=6$



180 pass

$$A_5(W) = 30W^2 + 270W + 350W^3$$

$$\left( (1-z)^{-1} A_5\left(\frac{1}{1-z}\right) \right)^{1/2} = 1 \quad \text{!!} \quad \frac{30}{10 \cdot 3} = 1$$

60 pass

$$\frac{270}{10 \cdot 6 \cdot 7} = \frac{9}{28}$$

$$\approx 0.3214$$

$K=5$

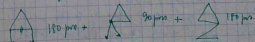


240 pass

60 pass

$$\frac{270 \text{ pass}}{10 \cdot 5 \cdot 6 \cdot 7} = \frac{270}{2100} = \frac{9}{70}$$

$$\frac{27}{42}$$



$$\text{CYCLE LENGTH} \quad \text{PB}(C_3=3) = \frac{2^9}{42} \quad \text{PB}(C_4=4) = \frac{11}{42} \quad \text{PB}(C_5=5) = \frac{2}{42}$$

(P)







# FIRST CYCLE IN EVOLVING GRAPHS

Sunday September 1<sup>st</sup>, 1985

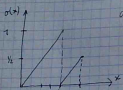
A SUNDAY'S DIVERTIMENTO:  
The distribution of  $\{ (3/2)^n \}$

After reading  
able but of  
Picard's book on  
Functional Equations

Problem is: (open) what is the distribution of  $\{ (3/2)^n \}$  in the sequence class on  $[0, 1]$ .

What should be the limiting distribution? If it admits a density  $\alpha(x)$

then how we are dealt with the iterates of  $\alpha(x)$



we should have because of "flow" conservation

$$\alpha(x) = \frac{2}{3} \alpha\left(\frac{1}{2}x\right) \text{ if } x > \frac{1}{2}$$

$$\alpha(x) = \frac{2}{3} \alpha\left(\frac{3}{2}x\right) - \frac{2}{3} \alpha\left(\frac{3}{2}x + \frac{1}{2}\right) \text{ if } x < \frac{1}{2}$$

Selon la repartition de  $U^{(n)}(x)$  on trouve pour  $n$  pair  $0-2^k/0.1-2^k/1, \dots$

	$n=500$	$n=1000$	$n=1500$	$n=2000$	$n=2500$	$n=3000$	$n=3500$
1	25	152	205	208	344	410	438
2	69	144	217	233	332	443	518
3	62	129	196	263	340	407	468
4	70	131	197	241	323	332	451
5	59	115	179	234	292	351	412
6	42	93	125	168	208	248	285
7	38	72	110	146	181	215	253
8	31	68	106	144	193	224	247
9	31	58	88	110	135	163	191
10	25	48	77	100	122	144	161

$$\int_0^1 \alpha(x) dx = 1$$

$$A(x) = A(2x/3) + A(x/2) - A(x/3) \text{ if } x > 1/2$$

$$A(x) = A(2x/3) + A(2x/3 + 1/2) - A(x/2) \text{ if } x < 1/2$$

les transformées de la distribution  
 $\int_0^1 \alpha(x) dx = 1$  etc.

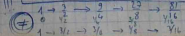
les transformées de la distribution  
est-ce que l'opération

Quelle est  
l'équation  
générale  
de ces 2 relations?

Les résultats lors de  
la nouvelle = 11  
NB - Différence de  $F$   
 $d = 10$  par un plan  
 $20\%$

la convergence pour  $f(x) = \frac{1}{3} f(2x/3) + \frac{1}{3} f(x/2) - \frac{1}{3} f(x/3)$  pour  $x = 1-10$

NB: on those things I have  
been making  
erroneously for a recurrence  
 $U_{n+1} = \left\{ \frac{2}{3} U_n \right\}$   
 $\neq \left\{ \left( \frac{2}{3} \right)^n \right\} !!$



# FIRST CYCLE IN EVOLVING GRAPHS

Another approach  $u_n$  has a distribution  $D$  over  $(0, 1)$   
 $u_0 = \left\{ \frac{1}{2} \right\}$   
 $\Rightarrow \cos(u_n \pi)$  has distribution  $\cos(\pi D)$  over  $(-1, 1)$

$\Pr\{u \in [x, x+dx]\} = \alpha(x) dx$   
 $\Rightarrow \Pr\{\cos \pi u \in [\cos(\pi x), \cos(\pi(x+dx))]\} = \alpha(x) dx$   
 $\Rightarrow \Pr\{Y \in [y, y+dy]\} = \frac{\alpha(\arccos(\frac{y}{\pi}))}{\pi \sqrt{1-y^2}}$   
 $y = \cos \pi x$   
 $x = \frac{1}{\pi} \arccos y$   
 $dy = -\pi \sin \pi x dx$   
 $= -\pi \sqrt{1-y^2} dx$

$\Rightarrow Y = \cos(\pi u_n)$  should have distribution with density  $\frac{d(\arccos(\frac{y}{\pi}))}{\pi \sqrt{1-y^2}}$

Set  $y_n = \cos(\pi(\frac{1}{2}))$ . Intensity find in that  $y_{n+1}$  can be (more or less) computed from  $y_n$ . **Arg, there is the rub!**

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta = \cos(3 \arccos(\frac{x}{2}))$   
 $\cos \theta = 2\cos^2 \frac{\theta}{2} - 1 \Rightarrow \cos \frac{\theta}{2} = \pm \sqrt{\frac{1+\cos\theta}{2}}$

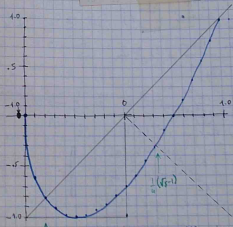
$y_{n+1} = \pm \sqrt{\frac{1+y_n}{2}} \left( \frac{2y_n}{2} \right)$

Let  $f(y) =$  with  $\oplus$  sign above. then we compute  $f^{(n)}(-2)$  and find for the histogram of iterates after  $n = 100, 1000, 1500$  iterations

-2	1	156	313	466
	2	91	181	273
	3	66	134	205
	4	33	67	118
	5	23	47	74
	6	28	54	76
	7	30	73	64
	8	16	36	65
	9	46	93	138
	10	-	-	-

Computation with 80 digits accuracy. Is it meaningful?  
 Question is of course: What is the effect of keeping the same sign all the time???

$f(y) = \sqrt{\frac{1+y}{2}} (2y-1)$



Maybe unless there exists a 0-1 sequence and that  $\left\{ (-1)^{\sum_{k=1}^n x_k} \right\}_{n \geq 1}$  DENSE  
 But it is not that enough???

NB: Keeping sign constant in  $f^{\circ}$  corresponds to iterating

$-0.99991 = -\sqrt{\frac{1+(-0.99991)}{2}}$



All this results of Ergodic Theory!

$\Rightarrow 3 \frac{y}{2}$  if  $3 \frac{y}{2} < 1$   
 $\Rightarrow -1 \left( \frac{3y}{2} - 1 \right) \Rightarrow 2 - \frac{3y}{2}$

using a normal  $\rightarrow$  relation  
 3/2  
**KOHELL**



# FIRST CYCLE IN EVOLVING GRAPHS

Histogram of  $\{S_j(n)\}$

Quite different from previous stuff!

$n = 500$  iterations from  $t = 0$   
 $n = 100$  iterations  $\rightarrow$

Another random idea - try to use Heuristics instead of theorem

$$S_j(n) = \frac{1}{n} \sum_{k=1}^n \epsilon_{j,k}$$

$$f_j: S_j(n) \rightarrow 0 \Rightarrow \text{Uniformly null}$$

$$\sum_{k=1}^n \epsilon_{j,k} \xrightarrow{\text{Mellin}} f(s) \sum_{k=1}^n u_k^{-s} \xrightarrow{\text{Mellin}} 0$$

Here give something like

$$\frac{1}{n} \left\{ \frac{1}{k} \right\}_{k=1}^{n+1} \xrightarrow{\text{Mellin}} \frac{1 - (n+1)^{-s}}{1 - (n+1)^{-s-1}} f(s) x^{-s} \rightarrow 0 \quad (n \rightarrow \infty)$$

Problem is (of course) that  $|f(1+i^r)| = O(e^{-\pi r/2})$

while  $(x^{-s})$  if  $x = e^{-\pi/2 + i \log(x)z}$

$$x^s = e^{-\pi/2 s + i \log(x) z s}$$

Perhaps: something CAN STILL BE DONE ?!???

Asymptotics

Loop functions

Un petit bout de théorie ergodique en passant (à l'après Brody-Davis).

Def: Soit une mesure  $m$  (pas forcément) sur  $\Omega$  tel que  $\exists$  premier  $h$  mesure  $\forall \epsilon \in \mathcal{E}$   
 $m(E) = m(\theta^h(E))$

Ex:  $G = (0, 1)$   $\theta(x) = x + 1/2$   $\theta(x) = 1 - x$   $\theta(x) = \{2x\} = 2x \text{ mod } 1$



$E = \{x \in [0, 1] \mid x \in [0, 1/2]\}$   
 "carré" ou "triangle" de 2 unités unitaires.

Théorème de Poincaré: Soit  $f$  point de  $\Omega$  est toujours récurrent, c-à-d.

$\forall \epsilon > 0 \exists N \in \mathbb{N}$   $\forall n > N$   $\exists k \in \mathbb{Z}$   $\theta^k(f) \in E$  est  $\infty$

Théorème de Birkhoff:  $f$  intégrable,  $\exists \bar{f}$  intégrable  
 $\frac{1}{n} \sum_{k=0}^{n-1} f(\theta^k(x)) \rightarrow \bar{f}(x)$  p.p.

$f$  est  $\theta$ -invariante, c-à-d  $\bar{f} = f \circ \theta$

$\int_A f \, dm = \int_A \bar{f} \, dm \quad \forall A$  qui soit invariant  $A = \theta^{-1}A$

Théorème de S. von Neumann  $f \in L^2$

$$\frac{1}{n} \sum_{k=0}^{n-1} f(\theta^k(x)) \rightarrow \int f \, dm \quad \text{p.p. et } \int f \, dm \text{ est } \theta\text{-invariant}$$

Rappel: Convergence  $\sum_{k=0}^{\infty} \frac{1}{2^k} = 1$

Idée de la dem. v.N.  $f \rightarrow \bar{f} = f \circ \theta$  ou meilleur  $\langle f, \theta^n f \rangle = \int f \theta^n f \, dm$

$f$  est linéaire et continue. Puis  $\exists$  de  $L^2$  2 normes pour  $\theta$ -invariantes

$\exists$  un  $\theta$  tel que  $\forall f, g \in L^2$   $\langle f, \theta^n g \rangle \rightarrow \langle f, g \rangle$  ou  $\exists$  un  $\theta$  qui est  $\theta$ -invariant

et  $\theta$  est en l'attente de nous pour le coup de  $I = \Pi(L^2)$ .

Idée de la dem. v.N. d'après Yosida-Kakutani

$f \in L^2$ ,  $E = \{x \in \Omega \mid \exists \epsilon > 0 \text{ tel que } \int_{\theta^n(x)} f \, dm \geq \epsilon \text{ pour } n \geq 0\}$

Applications (A) de type 1. Dim.  $R/\mathbb{Z}$   $f(x) = \cos(x)$

$\forall f \in L^1$   $\frac{1}{n} \sum_{k=0}^{n-1} f(x + k\alpha) = \int f \, dm$  (p.p.)

(B)  $f(x) = \cos(x)$  on considère les normes normales de  $\text{Boxcar}$

pour  $t$  tel on a  $\int_0^t f(x) \, dx = \frac{1}{2}$

Constat: Si les deux normes sont équivalentes, alors  $\int f \, dm = \int f \, dm$



# FIRST CYCLE IN EVOLVING GRAPHS

## RANDOM GRAPHS

Returning to original problem:

$$\text{Evaluate } \int_0^{\infty} \frac{\lambda^n}{n!} \left(1 + \frac{\lambda}{n}\right)^{-n} = \text{Int} \left( \frac{\lambda}{\lambda + \frac{1}{n}} \right) \times n!$$

Very exact (numerical) saddle point with  $\lambda$  & compare with exact value

Value of the integrand ~~at~~  $\lambda = 1$  of

$$n! \int_0^1 (1-u)^n e^{-\frac{1}{1-u}} du$$

$n=5$ :	0.31452	Note: there is a factor of $\frac{1}{n}$ in this integrand! But should be multiplied by $n!$ for normalization purposes
$n=10$ :	2.9195	
$n=15$ :	8.0334	
$n=20$ :	15.4929	
$n=30$ :	36.8734	
$n=40$ :	66.12112	

This is a bit change!  
(seems to grow too fast)

Let's summarize:

15 peaks (check 5th)

Expectation

$n=5$   
 $n=10$   
 $n=15$   
 $n=20$   
 $n=40$   
 $n=80$   
 $n=160$   
 $n=320$

Exact values	with saddle point (30 pts, by using $\frac{1}{n}$ )
0.33731	rather close
0.8989	close
1.2172	close
1.6760	close
2.1710	close
2.9195	close
8.0334	close
15.4929	close
36.8734	close
66.12112	close

Notes are confirmed with 15 peaks  
and 60 peaks

# peaks in  $n$  is  $n/30$

Conclusion: For  $n=40$ , the saddle pt approximation is not too inaccurate

From numerical data  $n=40$ : the profiles of both functions are quite similar (NB: exact ratio  $\frac{1}{n}$  div, exp. was  $\frac{1}{n}$ )  
 $\Rightarrow$  slight discrepancies to be expected.

Saddle pt. exact programme

The saddle pt curves & the exact curves, when normalized at  $\lambda=1$  show that:

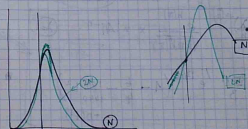
(A) - the  $\lambda$  s.t.  $\varphi_n$  is maxim tends to 1 (exact saddle)  
-  $\frac{\varphi(\lambda_{max})}{\varphi(1)} \rightarrow 1$

$\Rightarrow$  the curve becomes skinnier & skinnier!

(B) The value  $\varphi(\lambda_{max})$  increases (this is still confusing - it seems this happens in such a way that Area increases... as  $n \rightarrow \infty$  (of course!))

(C) The contribution to the integral comes from the area  $0..1 + \epsilon_n$   
 $1 + \epsilon_{10} = 3$ ;  $1 + \epsilon_{20} = 2.3$ ;  $1 + \epsilon_{40} = 1.95$ ;  $1 + \epsilon_{80} = 1.75$   
from numerical data of SAD. (read on the plot)

Evolution when normalized at  $\lambda=1$  gives



Spl 3/1955 Observation with saddle pt skinnier (large  $n$ )

$$\lambda = \frac{\varphi_{5120}}{\varphi_{1760}} = 1.4772 \quad \text{for saddle pt. skinnier.} \quad \frac{\varphi_{3760}}{\varphi_{1280}} = 1.50735$$

$$\frac{\varphi_{1720}}{\varphi_{640}} = 1.5348$$

NB: # of discretization points should adapt to  $n$ .



# FIRST CYCLE IN EVOLVING GRAPHS

The saddle point when  $\lambda=1$  is  $(1-\epsilon)$  with: **MAXV**

$$2n - n(1-\epsilon) - \frac{n}{1-\epsilon} + \frac{1}{\epsilon} = 0$$

$$\bullet h'(y) = my + n(y - y^2) - \ln \log y - \log(1-y)$$

$$\bullet h''(y) = 2n - ny - \frac{n}{y} + \frac{1}{1-y}$$

Sol:  $n\epsilon^2 = 1 - \epsilon$   
 $\epsilon = n^{-2/3}(1-\epsilon)^{1/3}$

$$\Rightarrow \epsilon = n^{-1/3} - \frac{1}{3}n^{-2/3} + 0 \cdot n^{-1} + \frac{1}{6}n^{-4/3} + \dots$$

$$\Rightarrow h(y) = e^{3/2 n^{1/3} + 1/3 - \log(n) + 1/6 t + 1/6 t^2 \dots} \quad \text{sol } (\epsilon = n^{-1/3})$$

$$h''(y) = \frac{1}{\epsilon^2} (3 + t + 7/3 t^2 + \dots)$$

$$\Rightarrow \frac{h(y)}{\sqrt{2\pi h''(y)}} = \frac{e^{1/3 - 3/2 n^{1/3}}}{\sqrt{6\epsilon}} \left(1 - \frac{t}{6} - \frac{107 t^2}{1640} - \frac{47 t^3}{5150} + \dots\right)$$

We find locally  $(\lambda=2)$   $(\epsilon = n^{-1/3})$

$$\left(1 + \frac{\lambda}{n}\right)^{-n} \frac{h(\sigma)}{\sqrt{2\pi h''(\sigma)}} \times n! \lambda^{n-1} \quad (\lambda=1)$$

BROWNIAN  
MAXV

$$= \sqrt{\frac{2\pi n}{3}} e^{1/3 - 3/2 n^{1/3}} \left(1 - \frac{t}{6} - \frac{107 t^2}{1640} + O(\epsilon^3)\right)$$

I find with these estimates (They should be a factor of  $1/2$ )

$$f_1(n=40) = 24.893$$

$$f_1(n=80) = 35.8717$$

$$f_1(n=160) = 51.4577$$

$$f_1(n=320) = 73.5708$$

$$f_1(n=640) = 104.923$$

$$f_1(n=1280) = 149.356$$

**SADP** (with smaller  $\lambda$ )  
and  $(1/\epsilon)$  factor

factor = 2.2615

$$9.32530$$

$$14.64677$$

$$20.86031803$$

$$27.07413708$$

EXACT SADP

ASYMPT. SADDLE  
0.5 (cos 200)

Thus  $\lambda$  in **MAXV** is relative to  $\int_0^{\infty} \frac{d\lambda}{(1+\lambda)^n} [\dots]$

Conclusion For  $\lambda=1$ , the saddle point is  $1 - n^{-1/3} + \frac{1}{3}n^{-2/3}$

The saddle point for general  $\lambda$

Equation is:

$$h = my + \frac{n}{\lambda} (y - y^2) - n \log y - \log(1-y)$$

$$h' = m - \frac{n}{\lambda} - \frac{n y}{\lambda} - \frac{n}{y} + \frac{1}{1-y}$$

$$= n(1 + \frac{1}{\lambda}) - \frac{n}{\lambda} y - \frac{n}{y} + \frac{1}{1-y}$$

Set  $y = 1 - \epsilon$ . Then:

$$n(1 + \frac{1}{\lambda}) - \frac{n}{\lambda}(1-\epsilon) - n(1+\epsilon) + \frac{1}{1-\epsilon} + \frac{1}{\epsilon} = 0$$

$$\frac{n}{\lambda} \epsilon - n\epsilon = \frac{n\epsilon^2}{1-\epsilon} + \frac{1}{\epsilon} = 0$$

$$n(\frac{1}{\lambda} - 1) \epsilon^2 - \frac{n\epsilon^3}{1-\epsilon} + 1 = 0$$

•  $\lambda > 1$  Solution is  $\frac{1}{\sqrt{n(1 - \frac{1}{\lambda})}} \approx \epsilon_{\text{sad}}$

•  $\lambda < 1$   $\epsilon \approx (1-\lambda) \Leftrightarrow \sigma \approx \frac{1}{\lambda}$ . So set  $y = \lambda(1-\rho)$  then find

$$m(1 + \frac{1}{\lambda}) - \frac{n}{\lambda}(1+\rho) - \frac{n}{\lambda}(1-\rho + \rho^2) + \frac{1}{(1-\lambda)(1-\rho)} + \frac{1}{1-\lambda} = 0$$

$$-n\rho + \frac{n}{\lambda}\rho + \frac{n}{\lambda} \frac{\rho^2}{1-\rho} + \frac{1}{1-\lambda} + \frac{\lambda\rho}{(1-\lambda)^2} + \frac{\lambda\rho^2}{(1-\lambda)^3} + \frac{\lambda\rho^3}{(1-\lambda)^4} = 0$$

$$\Rightarrow n\rho \left(\frac{1}{\lambda} - 1\right) + \frac{1}{1-\lambda} + \text{s.o.t.} = 0 \quad \rho \approx -\frac{2}{n}$$

$$\sigma = \lambda \left(1 - \frac{1}{n} + \text{s.o.t.}\right)$$



# FIRST CYCLE IN EVOLVING GRAPHS

Case  $\lambda > 1$  Sol  $1 - \frac{1}{\lambda} = K$  Eqn. 6a

$$-Kn\epsilon^2(1-\epsilon) - n\epsilon^3 + (1-\epsilon) = 0$$

$$-K\epsilon^2 + K\epsilon^3 - \epsilon^3 + \epsilon^2 - \epsilon\epsilon^2 = 0$$

$$\rightarrow K\epsilon^2 \approx \epsilon^2 \quad \epsilon \approx \frac{1}{\sqrt{n}} \frac{1}{\sqrt{1-\frac{1}{\lambda}}}$$

$$\epsilon = c_1 t + c_2 \epsilon^2 + c_3 \epsilon^3 + \dots$$

$$\frac{1}{\sqrt{n}} \quad \frac{1}{2\sqrt{n}} \quad -\frac{1}{2} K^{1/2} \frac{1}{\sqrt{n}} + \dots$$

$$\epsilon = \frac{1}{\sqrt{Kn}} = \frac{1}{2\sqrt{Kn}} + \left(\frac{5}{8} K^{3/2} - \frac{1}{2\sqrt{K}}\right) \frac{1}{\sqrt{Kn^3}} + \dots$$

$\sigma = 1 - \epsilon$  saddle pt.  
 $K = 1 - \frac{1}{\lambda}$

$$\begin{cases} h(\sigma) + \log t = \frac{1}{2} \log K + \frac{1}{2} + \frac{1}{2} \frac{K}{K^{1/2}} + \frac{1}{8} K - \frac{1}{2} Kn + O(t^2) \\ (2h'(\sigma) = 2K + \frac{3K}{\sqrt{K}} - \frac{3}{2} \frac{K}{K^2} \dots \end{cases}$$

Set  $t = \frac{1}{\sqrt{m}}$  & divide by  $n$   
 $K = 1 - \frac{1}{\lambda}$

1/2 exp(1/2) exp(1/2) ...  
 1/2 exp(1/2) exp(1/2) ...

Same as a h of  $\lambda$   
 $h(\lambda) = \frac{1}{2} \log \lambda - \lambda$   
 $\sim (1-\lambda)^2$  as  $\lambda \rightarrow 1$   
 $\sim -n(t-\lambda)^2$   
 $\epsilon$   
 NB This is allowed negative possible  $\lambda > 1$   
 $\log \lambda - \lambda + 1 = \frac{1}{2} \frac{(\lambda-1)^2}{\lambda}$   
 $\leq 0$

looks good

Good Tuesday 17:50 elapsed!

Next find  $\lambda \leq 1$  PC is relation between  $t$  and  $n$

It seems that  $t = n^{-1/2}$  trial program says it's not valid  
 $t = n^{-1/2}$  seems to work fine

Case  $\lambda < 1$   $t = \frac{1}{\sqrt{n}}$

1 find  $\frac{1}{2} + \frac{1}{2} t^2$   
 $SADU = t \frac{1}{2} \frac{1}{(1-\lambda)^{3/2}} e^{\frac{1}{2} + \frac{1}{2} t^2} \sqrt{2\pi n}$   
 $\frac{1}{\sqrt{n}}$   
 $\frac{1}{\sqrt{n}}$  partially because of saddle pt  
 $\times \frac{1}{2}$  because of scaling

$$\epsilon = \frac{\lambda}{(1-\lambda)^2} t = \frac{\lambda(1+t)}{(1-\lambda)^2} t^2 + \dots \text{ with } \sigma = \lambda(1-\epsilon)$$

$h(\sigma) = O(t^3)$  (with my expansion, looks good).

$$h(\sigma) = \frac{1}{\epsilon} \left( \frac{1}{2} + (1 - \log \lambda) \right) t^{-1} - \log(1-\lambda)$$

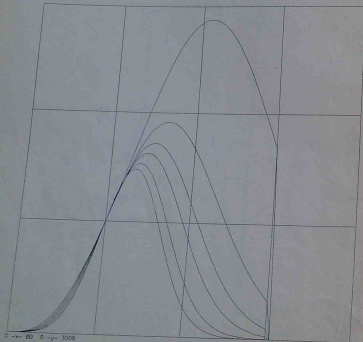
$$+ t \frac{1}{2} \frac{\lambda^2}{(1-\lambda)^3} + t^2 \frac{1}{6} \frac{\lambda^3(3\lambda+2)}{(1-\lambda)^6} + \dots$$

$K\epsilon^2 \gg \epsilon^3 \Rightarrow K \gg n^{-1/2}$   
 $\lambda \gg 1 + n^{-1/2}$



# FIRST CYCLE IN EVOLVING GRAPHS

Exod values:  $n=5, 10, 15, 20, 30, 40$



$$h^*(t) = \frac{1}{t} \cdot \frac{1}{\lambda^n (1-\lambda)} + \frac{\lambda+2}{\lambda(1-\lambda)^2} \Rightarrow t \frac{2\lambda^2 + 5\lambda - 1}{(1-\lambda)^2}$$

$$= \frac{1-\lambda}{\lambda^2 t} \left( 1 + \frac{\lambda(1+\lambda)t}{(1-\lambda)^2} + \frac{\lambda^2(2\lambda^2+5\lambda-1)}{(1-\lambda)^4} t^2 + \dots \right)$$

$\Rightarrow$  Everything is a function of  $\boxed{\frac{t}{(1-\lambda)^3}}$

- If for  $\lambda < 1$  anything is a fn of  $n(1-\lambda)^3$   
 $\Rightarrow$  behavior is  $\sqrt{n} \cdot n^{-1/3} = O(n^{1/6})$  | except if it tends to 0 for fixed  $\lambda \rightarrow O(1)$ !
- If for  $\lambda > 1$  anything is a fn of  $n(1-\lambda)^2$   
 $\Rightarrow$  behavior is  $\sqrt{n} \cdot n^{-1/2} = O(1)$ .

All this goes through checking...

WARNING Final expression is of the form

$$\boxed{CSE \lambda = 1 - \frac{h}{\sqrt[3]{n}}}$$

$$e^{h(\dots)} \quad e^{-\left(\frac{h}{\sqrt[3]{n}}\right) \log(\dots)}$$

Be careful one MAY NOT expand terms in  $t$  with a factor of  $n^{1/3}$  in the exponential at least...

$\Rightarrow$  check more carefully...

For instance:

$$n \log\left(1 + \frac{1}{n}\right) = n \left( \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \dots \right) = 1 - \frac{1}{2n} + \frac{1}{3n^2} - \dots$$

$$\boxed{n \log\left(1 + \frac{1}{n}\right) = \frac{\lambda n}{2} - \frac{1}{2} - \frac{1}{4} + O\left(\frac{1}{n}\right)}$$

But know that if  $\lambda = 1 - \mu t$   $t = n^{-1/3}$

$\lambda n = n - \mu n t \Rightarrow$  CAN'T expand in powers of  $\frac{t}{n}$   
 first substitute  $n t \rightarrow n^{2/3}$

Elegant solution would be: | Express all in terms of  $t$   
| Taylor expand  
| substitute for negative powers



# FIRST CYCLE IN EVOLVING GRAPHS

$$\tilde{x} = 1 - \mu v t, \quad (\lambda = 1 - \mu t)$$

$$m = 1/E^2, 3,$$

$$\text{Taylor } (-\mu \frac{t-1}{2}) \text{ by } (1 + \frac{t}{2}), t = 0, 3)$$

$$\text{Small } f = -\frac{1}{2} t^{-2} + \frac{1}{2} t t^{-2} + \frac{3}{4} + O(t)$$

At this stage (Wednesday 12.3.0 a.m.), I get ?

$$e^{-\mu n^2 + \frac{1}{2} \mu^2 + \frac{3}{2} \mu^3} - e^{\mu n} + \frac{1}{2} \mu n^{3/2} - \frac{1}{2} \mu^2 n^{3/2} + \frac{3}{4} \mu^3 n^{3/2}$$

$$x \frac{1}{(-\mu + 2\mu_1 + \frac{1}{\mu^2})^{3/2}} e^{-\mu \sqrt{2\pi n}} \times \text{Small } f \quad \left( \begin{array}{l} \text{Does not} \\ \text{Cancel} \end{array} \right)$$

$\frac{3/2}{\frac{1}{2} \mu + \frac{1}{2} \mu n + \frac{3}{4}}$

$$\text{h.o.f.} = -\frac{1}{2} \mu^2 + \frac{1}{2} \mu^2 + \frac{3}{2} \mu^3 = \log \mu, + \frac{1}{2} \mu^2 n^{3/2} + \frac{1}{2} \mu^2 n^{3/2} + \frac{3}{4} \mu^3$$

$$\text{h.o.f.} = (-\mu + 2\mu_1 + \frac{1}{\mu^2}) + O(t)$$

[& it a cancel  $\Rightarrow O(t^3)$  which is good!]

Saddle pt at  $1-E$  where

$$E = \mu t + \frac{(\mu+1)t^2}{(2\mu-3\mu_1)\mu}$$

↳ Grows like  $e^{8\mu t}$  when  $\mu=0, \mu \neq 1$   
 ↳  $\mu = 1-E = \frac{1}{3} t^2 + O(t^4)$   
 ↳  $\mu$  reaches  $\mu = 1$

Expansion when  $\lambda = 1 - \mu^{-1/3}$   
 $\lambda_1 t = O(t^2)$  ( $\lambda_1 t + \lambda_2 t^2 + \lambda_3 t^3 + \dots$ )

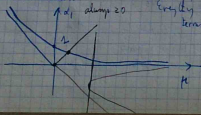
where  $\lambda_1$  is an algebraic function of degree 3 satisfying

$$\lambda_1^2 \mu - \lambda_1^3 + 1 = 0$$

whose behaviour is

$\mu, \lambda_1$  always 20

Explicitly is then expressed in terms of  $\lambda_1$ ...



(Problem may be with expansion)

Finally (out) I find (I forget the  $\lambda^n$  factor earlier)

$$\text{MAXV} = e^{-\frac{1}{2} \mu^2 + \frac{1}{2} \mu^2 - \frac{1}{2} \mu^2} \sqrt{e \pi n} e^{3/4} \left( x \frac{1}{\sqrt{2\pi n}} \right)$$

with again:

$$\text{MAXV} = \mu^3 = 1 + \mu^2 \mu$$

$$= \frac{1}{2} e^{-\frac{3}{4} + \frac{1}{2} \mu} e^{-\frac{1}{2} \mu^2 \mu + \frac{1}{2} \mu^3}$$

$$= \frac{1}{\sqrt{-\mu^2 + 2\mu_1 + 1}} e^{\frac{1}{2} \mu^3}$$

$$= \sqrt{3 + \mu^2}$$

$$\text{MAXV} = \frac{\sqrt{n}}{2} e^{-\frac{1}{2} \mu^2 \mu + \frac{1}{2} \mu^3} \int_0^{\infty} \dots d\lambda$$

Great!!!

$$\Rightarrow \text{Scale is } n^{-1/2} \quad \mu = n^{-1/3} d\lambda \Rightarrow E(x_n^*) = O(n^{1/6})$$

$\mu$  &  $\mu_1$  of  $\lambda_1$  alone this is:

$$\frac{1}{\sqrt{2 + \mu^2}}$$

part of continuation curve from  $x \rightarrow 2$   
 water out of  $\mu$   
 not all  $\mu$

$$\mu^3 - \mu^2 = 3 - 2\mu^3 - \frac{3}{\mu} + \frac{1}{\mu^2} + \frac{1}{\mu^3}$$

$$2 - \mu^2 = 2 - \mu^3$$

SIGN CHANGE

$$\frac{dx}{dy} = -1 - \frac{2}{y^3} \Rightarrow \text{integrated is}$$

$$\int_0^{\infty} \frac{e^{-\frac{1}{2}(3-2y^3 - \frac{3}{y^3} + \frac{1}{y^2})}}{\sqrt{2+y^3}} dy$$

not defined when  $y \rightarrow 0$   
 SINGULARITY at  $y = -2^{1/3} \dots$





# FIRST CYCLE IN EVOLVING GRAPHS

Conclusion: When  $\lambda \geq 1$   $\lambda = 1 - \mu n^{-1/3}$   
 everything works quite fine ( $\mu < 0$ )

SCALING IS GOOD!

Does not work when  $\lambda < 1$  i.e.  $\mu > 0$  since the previous integral diverges.

If one tries  $t = n^{-\alpha}$ , then one should make  $(u_i = \mu^{(1/\alpha)})$   
 $(a_i = 0)$

and  $e = \mu t + \frac{1}{2} t^2 + \dots$

but one gets finally  $\frac{e^{3/4}}{n}$

⇒ take a smaller  $t$  and let  $\mu \rightarrow \infty$  explicitly the contribution to come from a smaller order of  $t$  i.e.  $(1 - 0.999)^{10}$  etc.

changep = 1 or changep = 2

I find:  $e^{3/4} \sqrt{2n}$   
 too small

Equation for saddle point:  
 $t^2 = (u_i + u_i^2) \mu - u_i^3 - \mu$

3

Starting again

Everything seems to occur around

$u_i = 4 \pm 2^{1/3}$  which should give the value of PEAK.

namely  $(\lambda_{max} = -\frac{2}{3})^{1/3}$  and  $(1 - \lambda_{max} n^{-1/3}) = \lambda_{max}$

then agree well with saddle point (numerical) integrals:

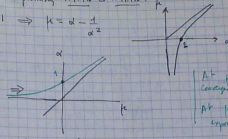
$\lambda = 4$	$n = 100$	$n = 400$	$n = 600$
4	0.0017		
3	0.06500		
2	0.3869	1.843203	
1	1.794305	5.3552	
0	4.0624	10.8156	
-1	5.06294	12.9412	
-2	5.1234	12.92347	
-3	5.1234	6.74335	
-4	1.0857		
-4	0.19352		
-10	1.12 · 10 <sup>-9</sup>		

Confirm numerically that the scaling factor is  $n^{-1/3}$

Also shows diff to be well controlled

Everything looks promising  $\frac{A_{\mu, \lambda, n}}{A_{\mu, \lambda, n}} \& \frac{A_{\mu, \lambda, n}}{A_{\mu, \lambda, n}}$

$$\mu a^2 = a^2 - 1 \Rightarrow \mu = a - \frac{1}{a^2}$$



At  $\mu = +\infty$  ( $a = 0$ )  
 Coefficient of the form  $k^{-3/2}$   
 At  $\mu = -\infty$  ( $a = 0$ )  
 exponential divergence

$a$  varies from 0 to  $\infty$   
 when  $\mu$  varies from  $-\infty$  to  $+\infty$

$$\mu^3 - a^3 = -3 + \frac{3}{a^3} - \frac{1}{a^6}$$

$$d\mu = (1 + \frac{3}{a^3}) da$$

⇒ Coeff is

$$\int_0^{\infty} e^{-3+3/a^3-1/a^6} \frac{(1+\frac{3}{a^3})}{\sqrt{2+a^3}} da$$

⇒ leads to

$$\int_0^{\infty} \frac{e^{-\mu t - t^2}}{\sqrt{t+t^2}} t^{1/3} dt \text{ as coefficient}$$

$\frac{\sqrt{6}}{Xn}$

can be split as  $\int_0^{\infty} e^{-t^2} (\sum a t^k) dt$

NB: A quick simulation for walking time (NB: we don't avoid deep water edges  $\alpha \rightarrow 1/2$ )

$n = 100$	100 realizations ave = 5.0415	223.32	231.8
$n = 500$	ave = 214.62	422.34	423.64
$n = 1000$	ave = 437.2		
$n = 2000$	ave = 870.70	860.12	846.98

Time is quite clearly 0.63 n



# FIRST CYCLE IN EVOLVING GRAPHS

Some first solutions

$n$	# Sim	Walking time	Galaxy	Convergence
100	100	54.99 (4%)	6.75	67.55
8400	100	Transition of memory after 29 simulation (av = 6.110)		
1000	1000	45.5 (10)	10.80	87.30
2000	10	90.20	3.90	86.30
6000	10	16.5 (3.0)	5.90	57.30
8000	10	34.0 (0)	12.60	26.70
1600	10	67.8	10.60	78.70
6400	25	2830.68	15.32	182.80
6400	25	2739.16	13.08	120.36
6400	25	2032.26	12.36	218.36
6400	25	2005.84	9.24	119.00
400	1000!	33.12 (3.4)	6.29 (active)	25.02 (approx)

avg = 13.25 m  
100.000  
x = 1000

Time for simulation:  $n = 6400$ : 3 seconds  $n = 10000$ : 0.5 sec.

40	500	23.63 (5.4)	5.10	15.26	
40	1000	13.56 (2.5)	5.19	15.29	6.17 (100)
20	1000	13.26 (5.4)	6.82	10.65	4.47 (60)
10	2000	2.55 (5.4)	5.37	6.77	3.37 (4)
5	2000	4.24 (5.4)	3.37	4.50	3.37 (1)

OH!!! Everything fits nicely with  $n^2$  formula

constants

## CYCLES IN A RANDOM GRAPH

6/9/85

### EVOLUTION MODEL (Kern)

Assumption: unvalued edges, no edge chosen twice, no "self-loops"

- (A) Let  $C_{n,k}[\Omega]$  be the # of configurations satisfying condition  $\Omega$ , then let  $p_n(\Omega)$  be the probability for  $\Omega$  to be satisfied. Then with  $N = \binom{n}{2}$ :

$$p_n(\Omega) = \sum_k \frac{(k-1)!}{N(N-1)\dots(N-k)} C_{n,k}[\Omega] \quad (1)$$

Same formula works for conditional expectations. Formula (1) gives the relation between the combinatorial model (counting) and the probabilistic model.

The  $C_{n,k}$  can be computed by the theory of the exponential generating function. For instance, if  $\Omega$  is a cycle has length  $c$ , then:

$$\sum C_{n,k} u^k \frac{z^n}{n!} = \frac{1}{z} \frac{y^c}{(1-y)} e^{\frac{1}{z}(y-y^2)} \quad (2a)$$

where  $y = y(zu)$  and  $y(t) = t e^{-y(t)}$

Thus for instance the expectation of  $\frac{1}{m}$  cycle length is under the probabilistic model is

$$\bar{K}_n = \sum_k \frac{(k-1)!}{N(N-1)\dots(N-k)} n! \left[ z^n \right] \left\{ \frac{1}{z} \frac{y^3}{(1-y)^2} e^{\frac{1}{z}(y-y^2)} \right\} \quad (2b)$$

- (B) Transformation (1) has an integral representation related to the Eulerian beta integral. Namely

$$p_n = \int_0^1 C\left(\frac{u}{1-u}\right) (1-u)^N du \quad C[u] = \left[ z^n \right] \sum C_{n,k} u^k \frac{z^n}{n!}$$

$\dagger$   $m$  no. dec,  $K$  edges.

# FIRST CYCLE IN EVOLVING GRAPHS

which becomes

$$p_n = n! \int_0^\infty \frac{v^n}{(1+v)^n} \left\{ \left[ \frac{1}{8} n \right] C\left(3, \frac{v}{1+v}\right) \right\} \frac{dv}{v(1+v)} \quad (3b)$$

and setting  $v = \lambda/n$ :

$$p_n = n! n^{-n} \int_0^\infty \frac{\lambda^n}{(1+\frac{\lambda}{n})^n} \left\{ \left[ \frac{1}{8} n \right] C\left(3, \frac{\lambda}{n}\right) \right\} \frac{d\lambda}{\lambda(1+\frac{\lambda}{n})} \quad (3c)$$

- ③ the idea is then to evaluate  $C\left(3, \frac{\lambda}{n}\right)$  numerically for fixed  $\lambda$  (actually only  $\lambda = O(1)$  matters somewhat) by

- (C1) The Cauchy integral (of course!)  
 (C2) the change of variables of the proof of Lagrange inversion theorem  
 (C3) Finally saddle point method.

For instance if we take as "parameter"  $\alpha$  the expectation of cycle length  $(-3)$ , by (C1)

$$R_n = n! n^{-n} \int_0^\infty \frac{\lambda^n}{(1+\frac{\lambda}{n})^n} \left\{ \frac{1}{2\pi i} \int_{\gamma} \frac{y^3}{(1-y)^2} e^{\frac{\alpha}{n}(y-y^2)} \frac{d\lambda}{\lambda(1+\frac{\lambda}{n})} \right\} \frac{d\lambda}{\lambda(1+\frac{\lambda}{n})} \quad (4)$$

$y = ze^t$

then by (C2)

$$R_n = n! n^{-n} \int_0^\infty \frac{\lambda^n}{(1+\frac{\lambda}{n})^n} \left\{ \frac{1}{2\pi i} \int_{\gamma} \frac{y^2}{(1-y)^2} e^{\frac{\alpha}{n}(y-y^2) + ny - n \log y} dy \right\} \frac{d\lambda}{\lambda(1+\frac{\lambda}{n})}$$

and applying (C3) represents some challenge.

- ① The saddle point method

What happens ultimately is: the integrand of (5) (ie the big function of  $\lambda$  and  $n$  in  $\int_0^\infty \dots d\lambda$ ) has a peak at  $\lambda = 1 + \epsilon_n$ .

For fixed  $\lambda$ , 2 cases appear

$$\lambda < 1, \quad \lambda > 1.$$

The saddle point  $\sigma = \sigma(\lambda, n)$  satisfies an algebraic equation of degree 3 and but the root to be picked changes suddenly at  $\lambda = 1$ .

For  $\lambda < 1$   $\sigma = \lambda + o(1)$ .

For  $\lambda > 1$   $\sigma = 1 - o(1)$ .

and using symbolic manipulation options, I find with  $I_n(\lambda)$  denoting the integrand of 5 normalized by  $n! n^{-n}$ :

$$\lambda < 1: I_n(\lambda) \sim \frac{\lambda^n}{(1-\lambda)^{3/2}} e^{n\lambda + \lambda^2/4} = O(1) \quad \text{for fixed } \lambda, n \rightarrow \infty$$

$$\lambda > 1: I_n(\lambda) \sim e^{n\lambda} \frac{1}{2} e^{-\lambda + \lambda^2/4} e^{n(\frac{1}{2} \log \lambda - \lambda)}$$

$\left\{ \begin{array}{l} \text{exponentially small} \\ \text{for fixed } \lambda, n \rightarrow \infty \end{array} \right.$

Thus the contribution should be localized around  $\lambda = 1$ .

At  $\lambda = 1$ :  $I_n(1) \sim \frac{1}{2} \sqrt{\frac{n}{3}} e^{n^{3/2}} = \text{increases as } O(\sqrt{n})$ .

thus the problem is to find how this behave for  $\lambda$  very close to 1 but a function of  $n$  itself.

- ② It takes a little while to find that the proper scaling factor is  $t = n^{-1/3}$  and we consider now

the saddle point  $\lambda = 1 - \mu t = 1 - \mu n^{-1/3}$  [fixed].

Then the saddle point  $\sigma = \sigma(\lambda, n)$  has an asymptotic expansion for  $t \rightarrow 0$  ( $n \rightarrow \infty$ ):

$$\sigma = 1 - d_1(t) + \frac{d_2 + d_1(t)}{2(d_1 - 3d_1(t))} t^2 + O(t^3) \dots \quad (6)$$

(more terms are necessary and have been computed using Maple).

In (6)  $d_1(t)$  is an algebraic function of degree 3 defined by

# FIRST CYCLE IN EVOLVING GRAPHS

$$d_1 k^2 - d_1^3 \neq 1 = 0 \quad (7)$$

$d_1$  increases continuously from 0 to  $+\infty$  as  $k$  goes from  $-\infty$  to  $+\infty$ .

Vary the saddle point method with expansion ~~done~~ expressed as functions of  $k$  and  $d_1 = d_1(k)$  (only implicitly defined), I find at some labour that:

$$I_n(\lambda) \approx \frac{\sqrt{n}}{2} e^{13/12} \frac{e^{-\frac{1}{6}(k^3 - d_1^3)}}{\sqrt{2 + d_1^2}} \quad (8)$$

thus after a rescaling nice  $d_1 = n^{-1/3} dk$ , from 8:

$$\bar{K}_n \sim \frac{e^{13/12}}{2} n^{1/6} \int_{-\infty}^{+\infty} \frac{e^{-\frac{1}{6}(k^3 - d_1^3)}}{\sqrt{2 + d_1^2}} dk \quad (9)$$

which can be re-expressed using  $k$  as an independent variable

theorem:

$$\bar{K}_n \sim \frac{e^{13/12}}{2} n^{1/6} \int_0^{\infty} \frac{e^{-3k - k^2}}{\sqrt{1+k}} k^{-1/3} dk \quad (10)$$

which is thus asymptotically proportional to  $n^{1/6}$ !

I find also with simulation programmes, for instance, I find that based on

$$\begin{aligned} 1000 \text{ simulations } n=1000 &\rightarrow \bar{K}_{1000} \approx 6.75 \\ 100 \text{ simulations } n=6400 &\rightarrow \bar{K}_{6400} \approx 13.25 \end{aligned}$$

Not too bad!...

Other applications: Lotka probability distribution of cycle length, size of cyclic component, waiting time etc... other models: replicated edges....

Summary Cycle length.

n	5	10	20	40	80	1000	500
Exact	3.3571 3.3571	3.5784 3.5784	4.4766 4.4766	5.17100			
Saddlept mean				5.9462	6.4517		
Simulated						6.29 6.46	6.7523 (277simul)
Asymp. $3 \cdot n^{1/6}$	3.92	4.40	4.94	5.54	6.22	6.46	6.46548

n	6400	32000		178600
Saddlept mean	14.691 14.691	19.500 19.500	24.60196 24.60196	28.720
Simulated	14.691	19.500		
Asymp. $3 \cdot n^{1/6}$	12.92	16.90		21.29

Simulation

M	100	6400	32000	178600	500
nsim	1000	1000	414	732	27200
K (nids) mean/std	6.415/ 3.4215	14.691/ 13.861	19.500/ 23.735	20.546/ 30.055	28.720 61
W (hms) mean/std	53.06/ 12.375	2870/ 223.02	14169/ 2626	14230/ 2519	23974 431
Component size mean/std	24.530/ 16.197	204.607/ 263.312	440.219/ 812.777	461.017/ 802.120	57.428 5

Other parameters: # of non-empty components.

$$\text{nice } \# \text{ comp} = n - W$$

Empirical: equal size graphs like  $25 \sqrt{n}$   
water size graphs like  $0.45 \cdot n$   
cycle graphs like  $3 \cdot n^{1/6}$

# of cycles of size  $k$   
→ need indicator to number of cycles  
→ already  
Jan 2014

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# FIRST CYCLE IN EVOLVING GRAPHS

(11)  $\frac{1}{2} \frac{R'(z_n)}{1-R(z_n)} e^{\frac{1}{2}(R(z_n) - R'(z_n)/z)}$

(12)  $k(z, u) = \frac{1}{2} \left( \frac{R(z)}{(1-R(z))^2} - R(z) - 2z \frac{R'(z)}{1-R(z)} \right) e^{\frac{1}{2}(R - R'/z)}$

(13)  $\frac{1}{n(n-1) - (n-k)z}$  pole of a given augmented configuration

(14)  $\frac{(k-1)!}{n(n-1) - (n-k)z}$  pole of a given augmented standard configuration

(15)  $\bar{Q}_n = \sum_k Q_{n,k} \pi_{n,k}$   
 coeff. of  $z^k$  under pole model

(16)  $\int_0^1 \frac{z^k}{(1-z)^k} (1-z)^k \frac{dz}{z}$

(17)  $\bar{Q}_n = \int_0^1 Q_n \left( \frac{z}{1-z} \right) (1-z)^n \frac{dz}{z}$  (check with sp)  $Q_n(z) = [z^n] \frac{R^n}{(1-R)^2}$

(18)  $\bar{Q}_n = n! \int_0^1 \frac{1}{(1-v)^n} Q_n(v) \frac{dv}{v(1+v)}$  other form for expectation

(19)  $\bar{X}_n = \int_0^1 \frac{v^n}{(1+v)^n} k_n(v) \frac{dv}{v(1+v)}$   $k_n(v) = [z^n] \frac{R^n}{(1-R)^2} = \frac{1}{2} \frac{R^n}{(1-R)^2}$

(20)  $k_n(v) = [z^n] \frac{R^n}{(1-R)^2} = \frac{1}{2} \frac{R^n}{(1-R)^2}$

(21)  $k_n(v) = \frac{1}{2n} \int_0^1 \frac{R^k(z)}{(1-R(z))^2} e^{\frac{1}{2}(R(z) - R'(z)/z)} \frac{dz}{z^{n+1}}$

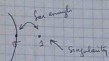
(22)  $k_n(v) = \frac{1}{2n} \int_0^1 \frac{z^k}{(1-z)^k} e^{ny + \frac{1}{2}(y - y/z)} \frac{dy}{y^n}$

lower  $\bar{K}_n = \int_0^1 \frac{z^n}{(1+z)^n} k_n\left(\frac{z}{1+z}\right) \frac{dz}{z(1+z)}$

(24)  $k_n\left(\frac{z}{1+z}\right) = \frac{1}{2n} \int_0^1 \frac{y^k}{(1-y)^k} e^{ny} \frac{dy}{y^n}$   
 $k_n(y) = \frac{1}{2} \log \frac{(1-y)^{n+1}}{(1-y-z)^{n+1}} - n \log y$

4/14/85 Observations on saddle point

$y = ze^{i\theta}$   $\bar{r}$  is saddle pt.  $\left| \frac{1}{1+z} e^{ny(1+z) - n/z} y^{-n} \right|$



$y = ze^{i\theta} \leq \frac{e^{-n}}{1-e^{-n}} |e^H|$

(a) ray of saddle pt.

$\text{Re}(H) \approx -n \frac{e^{2\theta}}{2} \left( (1 + \frac{1}{z}) - 2\sigma \right)$

okay, if  $\lambda < \lambda$  then this is always neg. and coeff is  $O(-n \lambda^2)$

$\Rightarrow$  take care to be  $O(n^{-k})$  & integrate between  $-\frac{e^{2\theta}}{\sqrt{n}}$  and  $+\frac{e^{2\theta}}{\sqrt{n}}$

(b) local expansion

$H = \log(1 - \frac{y}{z}) + n(1 + \frac{1}{z})y - ny/z - n \log y$

$H' = \frac{1}{1-y/z} + n(1 + \frac{1}{z}) - ny/z - \frac{n}{y}$

$H'' = \frac{1}{(1-y/z)^2} - n - \frac{n}{y^2}$

$H''' = \frac{2}{(1-y/z)^3} = \frac{2n}{y^3}$

$\sum_{j \geq 3} \frac{H^{(j)}(z=0)}{j!} (z=0)^j$

$\approx -\log(1 - \frac{z=0}{1-z=0}) + n \log(1 - \frac{z=0}{z=0})$

$\Rightarrow$  write the range explicitly in  $o(n)$ , I believe which is  $O(\frac{1}{\sqrt{n}})$   $\parallel$  Bal what about outside range.

lower  $\lambda_2 < \lambda < \lambda_1 < 1 \Rightarrow$   $v$ -poly in  $\lambda$  saddle point absolute value

$k_n\left(\frac{\lambda}{1+\lambda}\right) = O(\lambda)$   $\Delta$  but this is satisfied by  $n! n^{-n}$  (?)

$\lambda > 1$  Maybe take a contour around multiplicity where the bump is lower? (see p 96-101)

should be easier because of exponential decrease of overall thing.

outside the domain  $\text{Re}(z) < \frac{\log n}{\sqrt{n}}$  plus. But we have along  $y = ze^{i\theta}$   $\text{Re}\left( (1 + \frac{1}{z})y - \frac{1}{z} y^2 \right) < \text{same as } y = \lambda, \theta = \frac{1}{2}$

take  $y = \lambda e^{i\theta} \Rightarrow (\lambda+1) \cos \theta - \frac{1}{2} \cos 2\theta < 1 + \frac{1}{2}$   $x \geq 2$  and set  $\cos \theta = x$

Do we have for  $x \in (-1, 1)$ ?

$2(\lambda+1)x - \lambda(2x^2-1) < 2$   
 $2\lambda + \lambda(2x^2-1) - 2(\lambda+1)x > 0$  for  $x \in (-1, 1)$



# FIRST CYCLE IN EVOLVING GRAPHS

(26) saddle pt  $\mathcal{L}(y) = (1-y)(m(1+\frac{1}{2})y - \frac{1}{2}y^2 - m) + y = 0$

(27)  $\sigma = \lambda - \frac{\lambda}{(1-\lambda)^2} \epsilon + \frac{\lambda(1-\lambda)}{(1-\lambda)^3} \epsilon^2 + \dots \quad \epsilon = \frac{1}{m}$

(28)  $h_n(\lambda, \sigma) = \frac{1}{\lambda} \left( \frac{\lambda}{2} + 1 - \log \lambda \right) = \log(1-\lambda) + \frac{1}{2} \frac{\lambda^2}{(1-\lambda)^2} \epsilon + \dots$

(29)  $h_n^*(\lambda, \sigma) = \frac{1}{\lambda^2(1-\lambda)} \epsilon + \frac{\lambda+2}{2} \epsilon^2 + \dots$

(30)  $\frac{1}{\sqrt{2n}} \sum_{j=1}^n \epsilon_j = \sqrt{2n} \frac{e^{H(\epsilon)}}{\sqrt{H''(\epsilon)}}$   
 (under the hood of Taylor's theorem)  
 $\left\{ \begin{array}{l} \text{Converges when } |\log(\epsilon)| < \frac{\log 2n}{\sqrt{n}} \\ \& \quad |\delta| = 0 \end{array} \right.$

Donnerstag 29 September 1985 Probabilistic Counting & Antennas from a series of Brinckel

From Don Knuth's paper on "Informatics et Mathematiques"  
 Let  $H(p)$  be the Morse-Thue sequence.

$t(n) = (-1)^{H(n)}$   $v(n) = \# \text{ of ones in } n$

$t(n) = 1$	$n$	0	1	2	3	4	5	6	7	8	9	10
$t(2n) = t(n)$	$v(n)$	0	1	1	2	2	3	3	4	4	5	5
$t(2n+1) = -t(n)$	$t(n)$	1	-1	-1	1	1	-1	-1	1	1	-1	-1

Then:  $\frac{1}{\sqrt{2}} = \left(\frac{1}{2}\right)^{H(1)} \left(\frac{1}{4}\right)^{H(2)} \left(\frac{1}{8}\right)^{H(3)} \left(\frac{1}{16}\right)^{H(4)} \dots$

My naive comment from Probabilistic Counting involves (this is not really for you)

$$\varphi = \frac{2}{3} \prod_{p=1}^{\infty} \frac{(4p+1)(4p+2)}{(4p)(4p+3)} t(p) = \frac{2}{3} \left( \frac{5.6}{4.7} \right)^{5.6} \left( \frac{9.10}{8.11} \right)^{9.10} \dots$$

Then:  $\frac{1}{\sqrt{2}} = \frac{2}{3} \left( \frac{5.8}{6.7} \right)^{5.8} \left( \frac{9.12}{10.11} \right)^{9.12} \left( \frac{13.16}{14.15} \right)^{13.16} \dots$

$$\frac{1}{\sqrt{2}} = \frac{2}{3} \prod_{p=1}^{\infty} \frac{(4p+1)(4p+2)}{(4p)(4p+3)} \quad \text{checked numerically}$$

2/10/85 SUMMARY OF THINGS TO BE DONE

HAVE APPEARED SINCE 1985 Jan 1st

- |  |                                     |            |                    |
|--|-------------------------------------|------------|--------------------|
| 1. BIT '85   | Approximate counting                | FL         | for thesis - done  |
| 2. RAIRO 85  | Bubble memories                     | FL, Ch. Wu | 7 - RAIRO'S (done) |
| 3. IEEE-IC 85  | Analysis of packet                  | FL, Ho Ja  | 7 - IEEE-IC (done) |
| 4. IEEE-IC 85  | On any column                       | Tia, FL    |                    |
| 5. Annals of Math  | Algebraic methods for time analysis | FL, Re-So  |                    |
| <span style="border: 1px solid black; padding: 2px;">IN PRINT</span> |                                     |            |                    |
| 1. SIAM J. Comp.   | Registers                           | FL, Prod   | X (proofs)         |
| 2. JCSS  | Recursive Data Counting             | FL, Martin | X (proofs)         |
| 3. JACT  | Multidimensional                    | FL, Prod   | X (2nd paper)      |
| 4. SIAM J. Comp.   | Digital search trees                | FL, Seagle | X (proofs)         |
| 5. Buss  | Math. Techniques                    | FL         | X (Lecture Notes)  |

REVISED VERSIONS TO BE SENT (paper accepted)

- |                        |                           |                |                       |
|------------------------|---------------------------|----------------|-----------------------|
| 1. JACT                | Generating Estimation alg | Green, FL, Lad | X for done 8/85       |
| 2. J. of Mg            | Exponential bounds        | FL, Sahal      | X one 10/85           |
| 3. Inf. Sc. An Int. J. | Distributions             | FL, Pu-Vu      | X waiting around 2/85 |

SUBMITTED

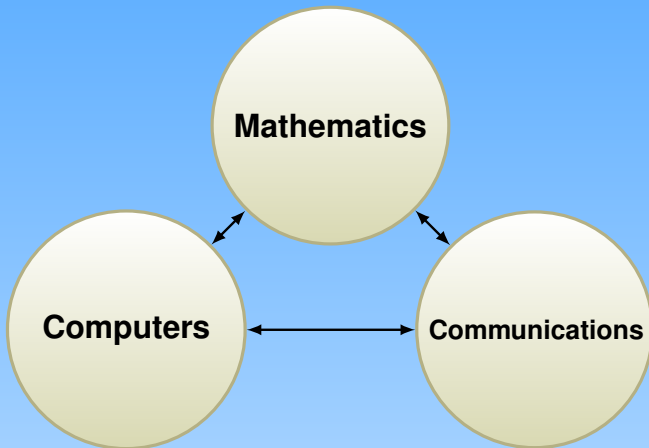
- |   |                    |               |                                       |
|---|--------------------|---------------|---------------------------------------|
| 1. <del>Algebraic Count.</del>  | ELLIPTIC functions | FL, Frazan    | not done<br>had offered<br>comp. 8/85 |
| <span style="border: 1px solid black; padding: 2px;">To be Submitted: bound editing required</span> |                    |               |                                       |
| 1. Elliptic Count   | Level # sequences  | FL, Prodinger | X offered                             |
| 2. Counting   | Adaptive Sampling  | FL            | X not 1/85<br>Comp. 8/85              |

To be done/finished

- |                                     |                        |               |                         |
|-------------------------------------|------------------------|---------------|-------------------------|
| 1. SIAM J. Comp (2)                 | 2 Stacks               | FL            | FCI July 85             |
| 2. J. of Graph Th. (2)              | Quies on random graph  | FL            | no Don Knuth            |
| 3. T.C.S                            | Asymptotic & trans     | FL            | offers Jan 83           |
| 4. Annals of Math                   | Relativ. to. coding    | FL, Re Seel   | delayed.                |
| 5. Algebraic Meth.                  | Functional Eqns        | FL            | polynomial 1/85         |
| 6. Annals of Math (2)               | Group Theory (Algebra) | FL            | No offered<br>No FCI 86 |
| 7. J. of Graph Th.                  | Quies on random graph  | FL            | no offered<br>no 1/85   |
| 8. <del>Counting</del> J. of Mg (2) | Quies on random graph  | FL, Prodinger | no offered<br>no 1/85   |



# SYNERGISTIC INTERACTION





# BACKHOUSE'S CONSTANT

## Radius of cv of

$$\frac{1}{1 + \sum_{k \geq 1} p_k z^k}$$

A030018 Coefficients in  $1/(1+P(x))$ , where  $P(x)$  is the generating function of the primes.

1, -2, 1, -1, 2, -3, 7, -10, 13, -21, 26, -33, 53, -80, 127, -193, 254, -355, 527, -764, 1149, -1699, 2436, -3563, 5133, -7352, 10819, -15863, 23162, -33887, 48969, -70936, 103571, -150715, 219844, -320973, 466641, -679232, 988627, -1437185, 2094446, -3052743 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS  $a(n+1)/a(n) \Rightarrow --1.4560749485826896714. - Zak Seidov, Oct 01 2011.$

## Backhouse's constant

From Wikipedia, the free encyclopedia

Backhouse's constant is a [mathematical constant](#) founded by N. Backhouse and is approximately 1.456 074 948.

It is defined by using the [power series](#) such that the [coefficients](#) of successive terms are the [prime numbers](#):

$$P(x) = 1 + \sum_{k=1}^{\infty} p_k x^k = 1 + 2x + 3x^2 + 5x^3 + 7x^4 + \dots$$

and where

$$Q(x) = \frac{1}{P(x)} = \sum_{k=0}^{\infty} q_k x^k.$$

Then:

$$\lim_{k \rightarrow \infty} \left| \frac{q_{k+1}}{q_k} \right| = 1.45607 \dots \text{ (sequence A072508 in OEIS).}$$

The limit was conjectured to exist by Backhouse which was later proved by P. Flajolet.

Binary 1.01110100110000010101001111101100...  
 Decimal 1.45607494858268967139959535111654...  
 Hexadecimal 1.74C153ECB002353B12A0E476D3ADD...

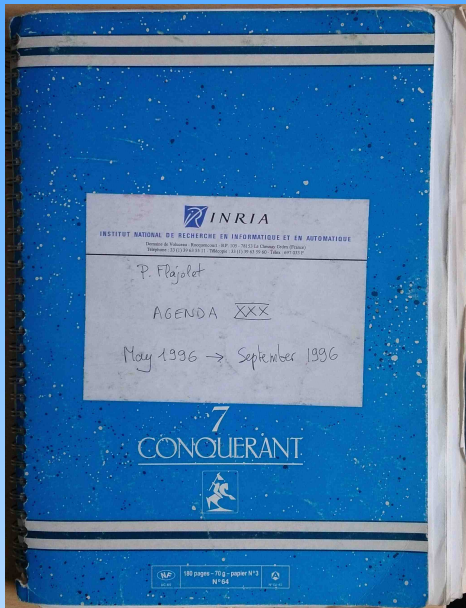
Continued fraction

$$1 + \frac{1}{2 + \frac{1}{5 + \frac{1}{5 + \frac{1}{4 + \dots}}}}$$

Note that this continued fraction is not periodic.



# BACKHOUSE'S CONSTANT



## Backhouse's Constant

Let  $P(x)$  be the formal power series whose  $n$ th term has coefficient equal to the  $n$ th prime number:

$$P(x) = \sum_{k=0}^{\infty} p_k x^k = 1 + 2x + 3x^2 + 5x^3 + 7x^4 + 11x^5 + 13x^6 + \dots$$

Let  $Q(x)$  be the formal power series defined by

$$P(x)Q(x) = 1$$

Thus  $Q(x)$  is the formal reciprocal of  $P(x)$  as a power series. Observe that this is pure formal algebra: no questions of analytical convergence are involved at all.

$Q(x)$  is an alternating series whose coefficients  $q_n$  are monotonically increasing in magnitude. Nigel Backhouse has observed that the ratios of successive coefficients tend to a certain constant, i.e., it appears that

$$\lim_{n \rightarrow \infty} \left| \frac{q_{n+1}}{q_n} \right| = 1.45607494858268967139959535111654356\dots$$

In a personal communication, Backhouse wrote:

The approximation given was generated in 37 seconds using Maple V (release 3) in batch mode on a Silicon Graphics Irix6.  $P(x)$  was taken to 550 terms and  $Q(x)$  produced as the Taylor series of  $P(x)^{-1}$ .

Unfortunately, I have no references to this result or anything like it. In particular, I have no evidence as to the originality of my observation. I was just curious, as someone with an amateur interest in number theory!

I should, of course, be very interested to hear, if, as a result of your enterprise, someone has anything to add to my rather thin story.

The 35-place decimal approximation above also appears at the CECM Inverse Symbolic Calculator web site. I am grateful to Simon Plouffe for pointing out to me the existence of this constant and to Nigel Backhouse for providing the information on which this essay is based.

Relevant Mathcad files will be included as time permits.



Return to the Favorite Mathematical Constants page .

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This page was updated November 21, 1995

# BACKHOUSE CONSTANT

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ON THE EXISTENCE AND THE COMPUTATION OF  
BACKHOUSE'S CONSTANT

Philippe Flajolet, Algorithms Project, INRIA  
November 25, 1995  
<Philippe.Flajolet@inria.fr>

## I. THE PROBLEM

Let  $p(n)$  be the  $n$ -th prime, with  $p(1)=2$ , and define  
infinity

$$P(x) = 1 + \sum_{n=1}^{\infty} \frac{p(n)}{x^n}$$

Nigel Backhouse examines the coefficients  $q(n)$  in the series  $Q(x)=1/P(x)$ :

$$Q(x) = \frac{1}{P(x)} = \sum_{n=0}^{\infty} q(n) x^n$$

We noticed empirically that the  $q(n)$  alternate in sign and that the ratio between successive values tends a constant equal (up to sign) to 1.45607... and called now "Backhouse's constant". See the description in Steven Finch's pages  
<<http://www.mathsoft.com/essolve/constant/backhouse/backhouse.html>>

## II. ANALYSIS

Here is what goes on. By the Prime Number Theorem, we have  $p(n) \sim n \log(n)$ , and at any rate  $p(n) < (n+1)^2$  for all  $n$ . Thus,  $P(x)$  is an analytic function in  $|x| < 1$ . Accordingly,  $Q(x)$  is meromorphic in  $|x| < 1$  and has only finitely many poles in any subdisk  $|x| < \rho < 1$  of the unit disk. Since  $P(0)=1$ ,  $Q(x)$  is analytic at 0. Thus, by Cauchy's coefficient formula,

$$q(n) = \frac{1}{2\pi i} \int_{\gamma} \frac{Q(x)}{x^{n+1}} dx$$

where the integration contour is a sufficiently small circle around 0. We observe that  $P(x)$  has a unique zero at  $s_0 = -0.486$  inside the disk of radius 0.75. Thus, integrating along  $|x|=0.75$  and taking into account the residue of  $Q(x)$  at  $z=s_0$  gives us

$$q(n) = \frac{1}{s_0^n p'(s_0)} s_0^n + O(75^{-n})$$

where  $s_0 = -1/s_0 = -1.45607$  is Backhouse's constant. This formula is quite good as its error term is of rate  $1/0.75^{1.33}$ , hence exponentially smaller than the dominant term.

It is possible to get farther by fishing for the next poles. In this way one can find better and better asymptotic expansions of the type

$$q(n) = c[0] s_0^n + c[1] s_1^{n+1} + c[2] s_2^{n+2} + \dots$$

(with suitable modifications if multiple poles were to be encountered), where a  $[i]=1/s[i]$  and  $s[i]$  is the  $i$ -th zero of  $P(x)$ . There doesn't seem to be real poles apart from  $s_0$

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nor multiple poles, but I have naturally no proof for this observation. Complex conjugate pairs of poles will make the correction terms fluctuating, as usual.

Note that for any collection of zeros given by a numerical process, a corresponding computer-assisted proof could be built. It suffices to use the principle of the argument (Henrici, Complex and Computational Analysis) to make sure that all zeros in a certain disk have indeed been captured. Also, all this proves that the coefficients  $q(n)$  eventually alternate in sign. This could be extended to all values of  $n$  by using constructive bounds in Cauchy's remainder integral and checking exhaustively the few dozen initial values not covered by the bounds.

Finally, there is a general theorem of Polyá-Carlson to the effect that a nonrational function with integer coefficients and radius of convergence 1 admits the unit circle has a natural boundary. Thus, as anticipated,  $P(x)$  and  $Q(x)$  both have the unit circle as a singular line.

## III. A GENERAL REMARK

What we just did is an instance of a general process well known in the analysis of coefficients of meromorphic functions. It is related to methods for coefficient asymptotics, like Darboux's method or singularity analysis, that are especially useful in "analytic combinatorics". An example that is close and that I like to use in teaching coefficient asymptotics is the following.

A composition of an integer  $n$  is a sequence of integers  $>0$  that add up to  $n$ . The number of compositions of  $n$  is  $2^{n-1}$ . Now consider compositions whose parts are restricted to be prime numbers 2,3,5,7,11,... How many are there?

Answer: about 0.303655263\*1.47228783<sup>n</sup>.  
Proof. Work with the series  $S(z)=1/(1-R(z))$   
where  $R(z)=z^2+z^3+z^5+z^7+z^{11}+\dots$

Philosophy: This discussion shows these questions to be infinitely easier than true arithmetical ones since this type of problem is (exponentially) oblivious of the fine structure of intervening analytic functions. For instance, one could define the "twin-Backhouse" constant by restricting  $P$  to terms corresponding to twin prime pairs! Even though we don't know a lot, we could still PROVE existence of the asymptotic form and COMPUTE the new constant to 1,000 digits in a matter of minutes. Similarly for the Fermat-Backhouse and the Mersenne-Backhouse constants!!!

Advertisement: A tutorial on these questions ("Complex Asymptotics and Generating Functions", INRIA Tech. Rep. 2026, Sept 1993) is available from <Philippe.Flajolet@inria.fr> and is going to be part of a forthcoming book by P. Flajolet and R. Sedgewick entitled "Analytic Combinatorics".

## IV. NUMERICAL VALUE OF BACKHOUSE'S CONSTANT

Here is in connection with Simon Plouffe's dictionary of real numbers  
<http://www.ccm.sfu.ca/projects/ISG.html>  
the value of Backhouse's constant to some 1300 Digits  
(determined in 4 minutes of CPU time with a Maple V.3 program on a DEC Alpha 3000 station.)

1.4560749485826896713995953511365435576531783748471315402707024\  
374140015062653898955996453194018603091099251436196347135486077\  
516491312121314292035177012831740536927499880254869237075808528



# BACKHOUSE CONSTANT

```

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45112405300017929785610674937708500577500543078199180068803215980\
62027263417356048148239430907193791269785500304113206689374270\
524605523103818234187452551243129272157858632005469559315813\
24650040902370866667117547152236564044351398169338973930393708\
45580836636739542046997815299374792625252091766965656321726658\
531118242706074521072844758644233717312597527697966190510539\
50647973036174806487308635116088714591201340018694939972951200\
31968555787957715446072017436793132019277084608142589327172752\
14035066947125582655125313554551262159917543249768704927031066\
824955171959738044474889305216942052481382787927915267956816\
962042960183918841574563549251600483940011902245484520213184\
607928204066710209464990039376979242935970760791495159294437\
90621403088414368576489094925101095437825265198368484856901017\
463899184591520397740466767628971155101327132174544437503346\
5950052270414125954003860725625514520109118577724099455296613\
69553185098749774020218534255771513121423357927183815991681750\
625176199614095578995402529309491627747326701699807286418966752\
89794574645089663967379786916133618184875;
    
```

## V. THE MAPLE PROGRAM

It is just Newton's method applied to  $P(x)$  with a close enough starting value, increasing the number of terms in the truncation of  $P(x)$  as we proceed. The call to  $b(n)$  gives 1357 exact digits in 4 minutes of CPU time.

```

Digits:=16; # must be there because of Maple's idiosyncrasy
b:=proc(n) # computes more than 10*n^2 digits of Backhouse's Constant
local ord, x, i, j, P, DP, t;
option remember;
timeprime:=proc(n) option remember; ithprime(n); end;
Digits:=15;
x:=1/.4560749468268967;
ord:=72;
for s from 1 to n do
  Digits:=2*Digits; ord:=2*ord;
  P:=1; DP:=0; xj:=x;
  for j from 1 to ord do
    t:=ithprime(j)*x; P:=P+t; DP:=DP+t*j; xj:=xj*x;
  od;
  x:=x^P*x/DP;
od;
RETURN(=1/x);
end;
b(6); # gives 678 exact digits in 70 seconds
b(7); # gives 1357 exact digits in < 4 minutes
    
```

$P(x) = b(x) - 10^{-n}$   
 $b(x) - 6(x) = 10^{-16}$   
 $b(2) - b(x) = 10^{-32}$   
 $b(x) - b(3) = 10^{-65}$   
 $b(x) - b(x) = 10^{-131}$   
 $b(x) - b(x) = 10^{-262}$   
 $b(x) - b(x) = 10^{-524}$

4 min  
 (the less than 10^4 digits of accuracy)

$10^{-16}$   
 $10^{-32}$   
 $10^{-64}$   
 $10^{-128}$   
 $10^{-256}$   
 $10^{-512}$

7.5 seconds  
 takes +70 seconds  
 takes +200 seconds

with better starting value

Let  $p(n)$  be the  $n$ -th prime ( $p(1) = 2$ ). By the ~~density~~ <sup>prim #</sup> theorem,  $p(n) \sim n \log n$  and  $p(n) < (n \log n)^2$  for all  $n$ . Thus  $P(z)$  is an analytic function in  $|z| < 1$ . In particular, in any disk of radius  $1 - \epsilon$ , it has only finitely many zeros. Accordingly,  $Q(z) = 1/P(z)$  is meromorphic in  $|z| < 1$  and has only finitely many poles in  $|z| \leq 1 - \epsilon$ .

Since  $P(0) \neq 0$ ,  $Q(z)$  is analytic at 0. Thus by Cauchy's formula we get  $q(n) = \text{Coeff}[z^n] Q(z)$  as

$$q(n) = \frac{1}{2\pi i} \int_{\gamma} Q(z) \frac{dz}{z^{n+1}}$$
 The integral being taken along circle  $\gamma$  sufficiently small ~~near~~ <sup>if we</sup> around the origin ~~to~~ <sup>to</sup> enlarge this circle to become  $|z| = 0.8$ . We observe that  $P(z)$  has a zero at  $z = 0.6 \dots$  inside  $|z| \leq 0.8$ . Thus ~~the~~ <sup>if we</sup> changing the contour to  $|z| = 0.8$  and taking the residue ~~into~~ <sup>the</sup> of the pole into account, we get

$$q(n) = \frac{1}{5L^2(s)} s^{-n} + O(R^{-n}).$$



# GRAY CODE FUNCTION

$$\sum_{0 \leq k \leq n} \binom{n}{k} (-1)^{n-k} g_k, \quad g_n = 2g_{\lfloor \frac{n}{2} \rfloor} + \frac{1 - (-1)^{\lceil \frac{n}{2} \rceil}}{2}$$

12/14/94.

On the Gray code function

A problem of Francis Clarke.

Let  $g(n)$  be the binary value of the  $n$ -th Gray code.

$n = 0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10$

$(g)_2 = 0 \ 1, 11, 10, 110, 111, 101, 100, 1100, 1101, 1111, \dots$

$g(n) = 0, 1, 3, 2, 6, 7, 5, 4, 12, 13, 17, \dots$

then

$$G(z) := \sum g(n) z^n$$

$$G(z) = \frac{1}{1-z} \sum_{k=0}^{\infty} 2^k \frac{z^{2^k}}{1-z^{2^{k+1}}}$$

The interpolation coefficients

$$a_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} g_k$$

are	$n$	0	1	2	3	4	5	6	7
	$a_n$	0	1	1	-4	12	-38	52	-80

but (!)  $a_{13}$  is the 1st exception being positive ...

$\frac{a_n}{(-2)^n}$  oscillates with a periodicity between primes of 2

It seems that (maybe!)  $\frac{a_n}{(-2)^n} \approx \log_2 n \cdot I(\log_2 n)$  (?)

We have

$$A(z) = \frac{1}{1+z} G\left(\frac{z}{1+z}\right)$$

Then the ~~zeros~~ singularities of  $A(z)$  accumulate vertically near  $\frac{1}{2}$  (up to  $\infty$ )

$$\frac{z}{1-z} \quad \frac{z}{1+z} \quad \omega^{2^k} z = z$$

Related question: What is  $\sum_{n=0}^{\infty} z^n (1 + \frac{z}{2^n})^{-n}$  what happens???

Naive approach: - Go to Euler transform

- Get a partial fraction decomposition

- Approximate the denominators

- Use Mellin transform to get the right form asymptotic form

$$\sum_{n=0}^{\infty} z^n \frac{z^{2^n}}{1+z^{2^{n+1}}} \quad \omega^{2^k} z = z$$

$$\rightarrow \frac{1}{2} \sum \frac{\omega}{\delta - \omega} \quad \text{Go Euler transform} \quad \sum \frac{\omega}{\frac{\delta}{2} - \omega}$$

$$\rightarrow \Gamma(\frac{1}{2}) \sum \frac{1}{1 - \delta^{1/2} \omega} \rightarrow \sum \frac{1}{1 - \delta^{1/2} \omega} = \sum \frac{(1-\omega)^{-1/2}}{\omega}$$

$$\rightarrow \frac{\Gamma_n}{\Gamma(-2)} = \sum \left(\frac{1-\omega}{2\omega}\right)^{-n} = \sum \left(\frac{1-\omega}{2\omega}\right)^n = \sum e^{-n \log_2 \frac{1-\omega}{\omega}}$$

Take real part  $\sum \cos \frac{\pi}{2} n \omega^n$

Maximum is near  $\theta = \frac{\pi}{2} + \epsilon \rightarrow$  ~~near the peak~~ near the peak ...

$$\sum e^{-\nu \epsilon^2} \cos\left(\frac{\nu \pi}{2} + \nu \epsilon\right) \quad \text{This is like } \frac{1}{\sqrt{\nu}}$$

$$\Phi(\frac{1}{2}) = \sum (-1)^n e^{-\nu \epsilon^2} \cos \nu \epsilon \quad \text{by stirl of } \left(\frac{1}{2}\right)^n \text{ or about}$$

then do this for  $\epsilon = \frac{1}{2^{2^k}}, \frac{1}{2^k}, \frac{1}{2^k}, \dots$

This looks like  $\Phi(\frac{1}{2}), \Phi(\frac{1}{4}), \Phi(\frac{1}{8}), \dots \rightarrow \left(\frac{\Phi(\frac{1}{2})}{1-z^{-2}}\right)$

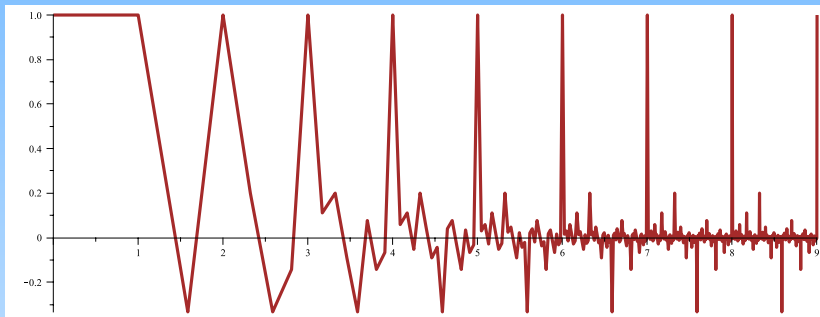
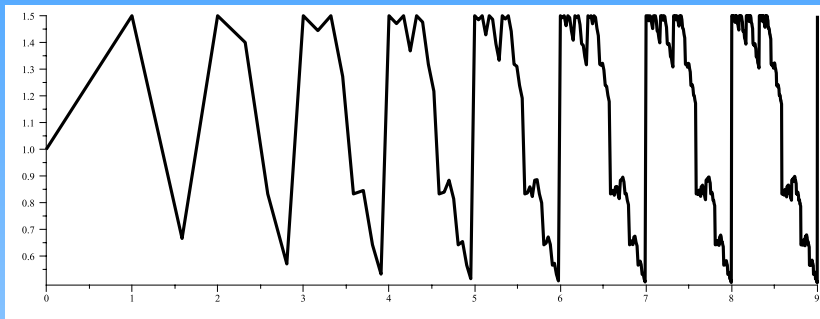
$$\Phi^*(z) = (1-z^{-2}) \Phi(z) \quad \text{this suggests } \log_2 z = \lfloor \log_2 n \rfloor$$

+ periodic function (!)

Can't be that easy!!!



# GRAY CODE FUNCTION: $g_n$ & $g_{n-1} - g_n$

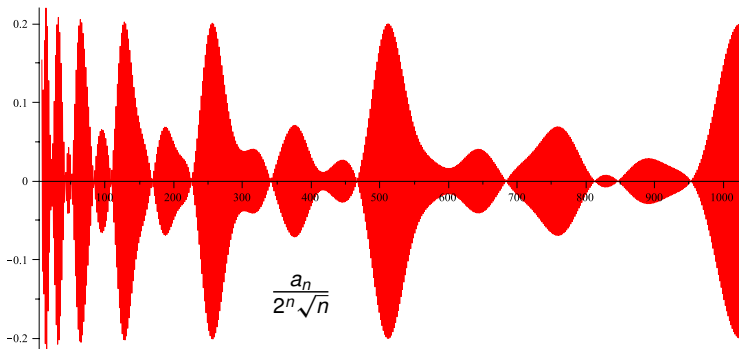


# GRAY CODE FUNCTION

$$a_n := \sum_{0 \leq k \leq n} \binom{n}{k} (-1)^{n-k} g(k)$$

$$\frac{a_n}{(-2)^n} = [z^n] G\left(-\frac{z}{2-z}\right) \quad G(z) := \sum_{j \geq 0} \frac{2^j z^{2^j}}{1 + z^{j+1}}$$

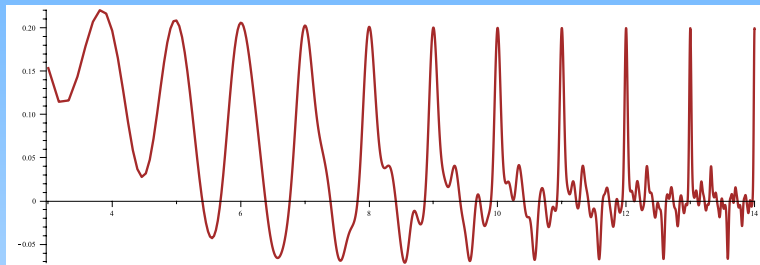
$n = 43$  is the first exception that  $a_n/(-2)^n > 0$



# GRAY CODE FUNCTION

$$a_n := \sum_{0 \leq k \leq n} \binom{n}{k} (-1)^{n-k} g(k)$$

$$\frac{a_n}{(-2)^n} = -\frac{1}{2\sqrt{2\pi n}} \int_{-\infty}^{\infty} e^{-(v-n)^2/(2n)} \sum_{3 \leq k \leq L_n+2} \frac{\sin\left(\frac{1}{2}v\pi\right)}{\cos\left(\frac{\pi v}{2^k}\right)} dv$$



**Asymptotics of  $\frac{a_n}{(-2)^n \sqrt{n}}$  remains open**





# ***OPEN PROBLEMS IN PF'S OEUVRES***



# INTRACTABLE

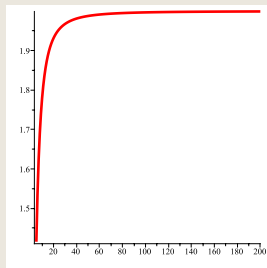
[PF44] (P762) & [PF98] (P217): ... *ce qui donne lieu à la plus célèbre conjecture de l'informatique*

**P  $\neq$  NP**

$$[\text{PF197}]: \left| \sum_{2 \leq k \leq n} \binom{n}{k} \frac{(-1)^k}{\zeta(k)} \right| = O\left(n^{\frac{1}{2} + \varepsilon}\right)$$

**$\equiv$  Riemann Hypothesis**

RH is also connected to algorithm complexity (with Vallée & Clement): [PF144] [PF157] [PF161]



# SOLVED

- [PF51] [PF69] [PF72] [PF108]: Height & diameter of BSTs (Devroye, Reed, Drmota, ...)
- [PF122]: height of quadrees (Devroye)
- [PF112] [PF152]: Quicksort limit law (Fill, Janson, Devroye, Neininger, ...)
- [PF147] Max deg in planar triangulations (Gao, Wormald)
- [PF200]  $(1 - \lambda)^{-\frac{1}{3}}$  realizable by stochastic context-free grammar? (Banderier, Drmota)
- ...



# SOLVED?

[PF114] [PF132] [PF144] (with Vallée)  
*Spectrum of the Euclid transfer operator*

Beginning around 1994,  
Underlying a series of papers  
Stating a set of conjectures  
seemingly proven in 2013 by Alkauskas

$$\mathbf{G}_s[f](x) := \sum_{m \geq 1} \frac{1}{(m+x)^{2s}} f\left(\frac{1}{m+x}\right)$$

Philippe was interested in computing the spectrum

- for  $s = 1$ : **Euclid algorithm**
- for  $s = 2$ : **Gauss reduction algorithm**



# CONJECTURES ON THE SPECTRUM OF $G_s$

For the Gauss-Kusmin-Wirsing operator  $G := G_1$

- All eigenvalues  $|\lambda_n|$  are **simple** & **strictly**  $\downarrow$
- They **alternate** in sign:  $(-1)^n \lambda_n > 0$
- $\lim_{n \rightarrow \infty} \frac{\lambda_n}{\lambda_{n+1}} = -\phi^2$  &  $\lambda_n \sim (-1)^{n+1} \phi^{-2n}$

Alkauskas announced in 2013 a proof of the conjectures

arXiv 1210.4083: “*In this work we prove an asymptotic formula for the eigenvalues of  $L$ . This settles, in a stronger form, the conjectures of D. Mayer and G. Roepstorff (1988), A. J. MacLeod (1993), Ph. Flajolet and B. Vallée (1995), ...*”

He asked Brigitte if other experiments were performed. Then with Julien, more computations made by Philippe were found ...



# STILL OPEN?

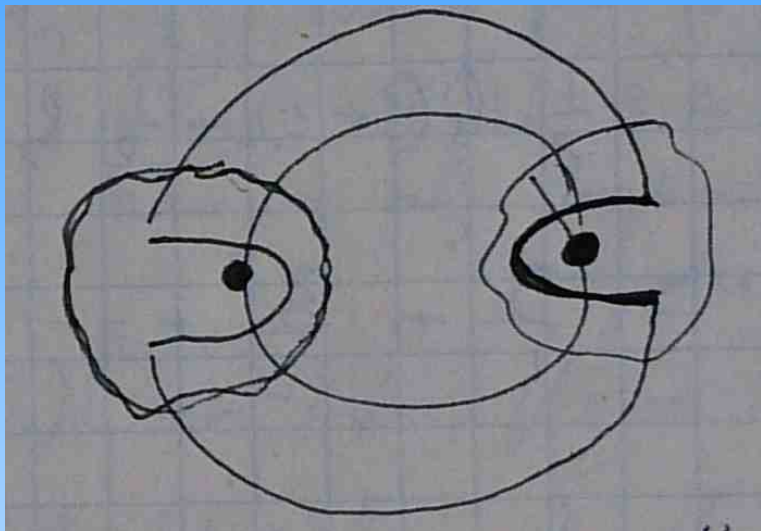
- [PF207]: non-holonomicity of  $\cos \sqrt{n}$ ,  $\cosh \sqrt{n}$
- [PF204]: *three-sided prudent polygons*,  $g_4 = 1 + \log_2 3$ ?
- [PF200]: Buffon machine for Euler's  $\gamma$ ?
- [PF191]: *hidden word statistics, convergence rate to normality?*
- [pF185]: Graeffe polynomials computable at a lower cost?
- [PF174]: *motif statistics under more general models?*
- [PF172]: robustness of interconnection in random graphs, finer properties like variance?



# STILL OPEN?

- [PF161]: trie statistics under general dynamic sources
- [PF157]: *continued fractions & comparison algorithms, many questions*
- [PF144]: continued fraction algorithms, uniformity of quasi-power for MGF?
- [PF69]: *linear worst-case time for tree-matching algorithms?*
- [PF64]: ambiguity of context free languages, many questions
- [PF54]: *collision resolution algorithms in random access systems, limit of stability?*







# WHAT TO DO WITH THE CAHIERS?

**Cahiers**



**Digital Forms**



**Web Accessible?**



**PolyPF  
Project?**

