

## ANALYSIS OF AN EXHAUSTIVE SEARCH ALGORITHM IN RANDOM GRAPHS AND THE $n^{c \log n}$ -ASYMPTOTICS\*

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**Abstract.** We analyze the cost used by a naive exhaustive search algorithm for finding a maximum independent set in random graphs under the usual  $\mathcal{G}_{n,p}$ -model where each possible edge appears independently with the same probability  $p$ . The expected cost turns out to be of the less common asymptotic order  $n^{c \log n}$ , which we explore from several different perspectives. Also we collect many instances where such an order appears, from algorithmics to analysis, from probability to algebra. The limiting distribution of the cost required by the algorithm under a purely idealized random model is proved to be normal. The approach we develop is of some generality and is amenable for other graph algorithms.

**Key words.** asymptotic expansion, random graphs, graph algorithms, generating functions, Laplace transform, saddle-point method, pantograph equation

**AMS subject classifications.** 60C05, 05A16, 05C69, 05C80, 68W40, 68R10

**DOI.** 10.1137/130916357

**1. Introduction.** An *independent set* or *stable set* of a graph  $G$  is a subset of vertices in  $G$  no two of which are adjacent. The *maximum independent set (MIS) problem* consists of finding an independent set with the largest cardinality; it is among the first known NP-hard problems and has become a fundamental, representative, prototype instance of combinatorial optimization and computational complexity; see [28]. A large number of algorithms (exact or approximate, deterministic or randomized), as well as many applications, have been studied in the literature; see [6, 26, 69] and the references therein for more information.

The fact that there exist several problems that are essentially equivalent (including maximum clique and minimum node cover) adds particularly further dimensions to the algorithmic aspects and structural richness of the problem. One of the most transparent ways of computing  $\alpha(G)$ , the cardinality of an MIS of  $G$  (or the stability number), is the following formulation in terms of polynomial optimization (see [1, 32]):

$$\alpha(G) = \max_{(x_1, \dots, x_n) \in [0,1]^n} \left( \sum_{1 \leq i \leq n} x_i - \sum_{(i,j) \in E} x_i x_j \right),$$

where  $E$  is the set of edges of  $G$ . Such an expression leads readily to an easily coded algorithm, but with deterministic exponential complexity  $O(2^n)$ . The algorithmic,

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\*Received by the editors April 9, 2013; accepted for publication (in revised form) December 5, 2013; published electronically March 6, 2014.

<http://www.siam.org/journals/sidma/28-1/91635.html>

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theoretical, and practical connections of many other formulations similar to this one have also been widely discussed; see [1].

Another simple means to find an MIS of a graph  $G$  is the following exhaustive (or branching or enumerative) algorithm. Start with any node, say,  $v$  in  $G$ . Then either  $v$  is in an MIS or it is not. This leads to the recursive decomposition

$$(1.1) \quad \alpha(G) = \max \left\{ \underbrace{\alpha(G \setminus \{v\})}_{v \notin \text{MIS}(G)}, \underbrace{1 + \alpha(G \setminus N^*(v))}_{v \in \text{MIS}(G)} \right\},$$

where  $\text{MIS}(G)$  denotes an MIS of  $G$  and  $N^*(v)$  denotes the union of  $v$  and all its neighbors. Such a simple procedure was the origin of many refined algorithms in the literature, including alternative formulations such as backtracking (see [68]) or branch and bound (see [26]).

Tarjan and Trojanowski [64] proposed an improved exhaustive algorithm with worst-case time complexity  $O(2^{n/3})$ . Their paper was followed and refined by many since then; see [6, 69] and [26] for more information and references. In particular, Chvátal [12] generalized Tarjan and Trojanowski’s algorithm and showed *inter alia* that for almost all graphs with  $n$  nodes, a special class of algorithms (which he called order-driven) has time bound  $O(n^{c_0 \log n + 2})$ , where  $c_0 := 2/\log 2$ . He also characterized exponential algorithms and conjectured that a similar bound of the form  $O(n^{c \log n})$  holds for a wider class of recursive algorithms for some  $c > 0$ . Pittel [56] then refined Chvátal’s bounds by showing that, under the usual  $\mathcal{G}_{n,p}$ -model (namely, each pair of nodes has the same probability  $p \in (0, 1)$  of being connected by an edge, and one independent of the others), the cost of Chvátal’s algorithms (called  $f$ -driven, more general than order-driven) is bounded between  $n^{(\frac{1}{4} - \varepsilon) \log_\kappa n}$  and  $n^{(\frac{1}{2} + \varepsilon) \log_\kappa n}$  with high probability for any  $\varepsilon > 0$ , where  $q := 1 - p$  and  $\kappa := 1/q$ .

The infrequent scale  $n^{c \log n} = e^{c(\log n)^2}$  is central to our study here and can be seen through several different angles that will be examined in the following paragraphs. The simplest algorithmic connection to the MIS problem is via the following argument. It is well known that for any random graph  $G$  (under the  $\mathcal{G}_{n,p}$ -model), the value of  $\alpha(G)$  is highly concentrated for fixed  $p \in (0, 1)$ , namely, there exists a sequence  $m_n$  such that  $\alpha(G) = m_n$  or  $\alpha(G) = m_n + 1$  with high probability; see [5]. Asymptotically ( $\kappa := 1/q$ ),

$$m_n = 2 \log_\kappa n - 2 \log_\kappa \log_\kappa n + O(1).$$

For more information on this and related estimates, see [5] and the references therein. Thus a simple randomized (approximate) MIS-finding algorithm consists of examining all possible

$$\binom{n}{m_n} + \binom{n}{m_n + 1} = O(n^{2 \log_\kappa n})$$

subsets and determining if at least one of them is independent; otherwise (which happens with very small probability; see [5]), we resort to exhaustive algorithms such as that discussed in this paper.

From a different algorithmic viewpoint, Jerrum [40] studied the following metropolis algorithm for maximum clique. Sequentially increase the clique, say,  $K$ , by (i) choosing a vertex  $v$  uniformly at random; (ii) if  $v \notin K$  and  $v$  is connected to every

vertex of  $K$ , then add  $v$  to  $K$ ; (iii) if  $v \in K$ , then  $v$  is subtracted from  $K$  with probability  $\Lambda^{-1}$ . He proved that for all  $\Lambda \geq 1$ , there exists an initial state from which the expected time for the metropolis process to reach a clique of size at least  $(1+\varepsilon) \log_\kappa(pn)$  exceeds  $n^{\Omega(\log pn)}$ . See [13] for an account of more recent developments on the complexity of the MIS problem.

We aim in this paper at a more precise analysis of the cost used by the simple recursive, exhaustive algorithm implied by (1.1). The exact details of the algorithm matter less, and the overall cost is dominated by the total number of recursive calls, denoted by  $X_n$ , which is a random variable under the same  $\mathcal{G}_{n,p}$ -model. Then the mean value  $\mu_n := \mathbb{E}(X_n)$  satisfies

$$(1.2) \quad \mu_n = \underbrace{\mu_{n-1}}_{v \notin \text{MIS}(G)} + \underbrace{\sum_{\substack{0 \leq k < n \\ v \in \text{MIS}(G)}} \pi_{n,k} \mu_k}_{v \in \text{MIS}(G)}$$

for  $n \geq 2$  with the initial conditions  $\mu_0 = 0$  and  $\mu_1 = 1$ , where

$$\pi_{n,k} := \mathbb{P}(v \text{ has } n - 1 - k \text{ neighbors}) = \binom{n-1}{k} p^{n-1-k} q^k.$$

How fast does  $\mu_n$  grow as a function of  $n$ ? (i) If  $p$  is close to 1, then the graph is very dense and thus the sum in (1.2) is small (many nodes being removed), so we expect a polynomial time bound by simple iteration. (ii) If  $p$  is sufficiently small, then the second term is large, and we expect an exponential time bound. (iii) What happens for  $p$  in between? In this case the asymptotics of  $\mu_n$  turns out to be nontrivial and we will show that

$$(1.3) \quad \log \mu_n = \frac{\left(\log \frac{n}{\log_\kappa n}\right)^2}{2 \log \kappa} + \left(\frac{1}{2} + \frac{1}{\log \kappa}\right) \log n - \log \log n + P_0\left(\log_\kappa \frac{n}{\log_\kappa n}\right) + o(1),$$

where  $P_0(t)$  is a bounded, periodic function of period 1. We will give a precise expression for  $P_0$ . Note that

$$(1.4) \quad \frac{\mu_n}{n^{\frac{1}{2} \log_\kappa n}} \asymp \frac{(\log n)^{\frac{1}{2} \log_\kappa \log n - 1 - \frac{\log \log \kappa}{\log \kappa}}}{n^{\log_\kappa \log n - \frac{1}{2} - \frac{1}{\log \kappa} - \frac{\log \log \kappa}{\log \kappa}}} \ll n^{-K} \rightarrow 0$$

for any  $K > 0$ , where the symbol  $a_n \asymp b_n$  means that  $a_n$  and  $b_n$  are asymptotically of the same order. Thus  $\mu_n = o(n^{\frac{1}{2} \log_\kappa n - K})$ . On the other hand, the asymptotic pattern (1.3) is to some extent generic, as we will see below.

An intuitive way to see why we have the asymptotic form (1.3) for  $\log \mu_n$  is to look at the simpler functional equation

$$(1.5) \quad \nu(x) = \nu(x-1) + \nu(qx),$$

since the binomial distribution is highly concentrated around its mean value  $pn$ , and we roughly expect that  $\mu_n \approx \nu(n)$ . This functional equation and the like (such as  $\nu_n = \nu_{n-1} + \nu_{\lfloor qn \rfloor}$ ) have a rich literature. Most of them are connected to special integer partitions; important pointers are provided in the Encyclopedia of Integer Sequences; see, for example, <http://oeis.org/A000123>, <http://oeis.org/A002577>, <http://oeis.org/A005704>, <http://oeis.org/A005705>, and <http://oeis.org/A005706>. In

particular, it is connected to partitions of integers into powers of  $\kappa = 1/q \geq 2$  when  $\kappa$  is a positive integer; see [15, 27, 50]. It is known that (under suitable initial conditions)

(1.6)

$$\log \nu(x) = \frac{\left(\log \frac{x}{\log_\kappa x}\right)^2}{2 \log \kappa} + \left(\frac{1}{2} + \frac{1}{\log \kappa}\right) \log x - \log \log x + P_1\left(\log_\kappa \frac{x}{\log_\kappa x}\right) + o(1)$$

for large  $x$ , where  $P_1(t)$  is a bounded 1-periodic function; see [15, 21]. Thus

$$|\log \mu_n - \log \nu(n)| = \left|P_0\left(\log_\kappa \frac{x}{\log_\kappa x}\right) - P_1\left(\log_\kappa \frac{x}{\log_\kappa x}\right)\right| + o(1).$$

We see that approximating the binomial distribution in (1.2) by its mean value

$$\mathbb{E}(\mu_{n-1} - \text{Binom}(n-1; p)) \approx \mu_{n-1} - \mathbb{E}(\text{Binom}(n-1; p)) \approx \mu_{\lfloor qn \rfloor}$$

gives a very precise estimate, where  $\text{Binom}(n-1; p)$  denotes a binomial distribution with parameters  $n-1$  and  $p$ .

An even simpler way to see the dominant order  $x^{c \log x}$  is to approximate (1.5) by the *delay differential equation* (since  $\nu(x) - \nu(x-1) \approx \nu'(x)$  for large  $x$ )

$$(1.7) \quad \omega'(x) = \omega(qx),$$

which is a special case of the so-called pantograph equations

$$\omega'(x) = a\omega(qx) + b\omega(x),$$

originally arising from the study of current collection systems for electric locomotives; see [38, 44, 53]. Since the usual polynomial or exponential functions fail to satisfy (1.7), we try instead a solution of the form  $\omega(x) = x^{c \log x}$ ; then  $c$  should be chosen to satisfy the equation

$$x^{1-2c \log \kappa} = 2ce^{c(\log \kappa)^2} \log x.$$

So we should take  $c = 1/(2 \log \kappa) + O(x^{-1} \log x)$ . This gives the dominant term  $\frac{(\log x)^2}{2 \log \kappa}$  for  $\log \omega(x)$ . More precise asymptotic solutions are thoroughly discussed in [16, 17, 44]. In particular, all solutions of the equation  $\omega'(x) = a\omega(qx)$  with  $a > 0$  satisfy

$$\begin{aligned} \log \omega(x) &= \frac{\left(\log \frac{x}{\log_\kappa x}\right)^2}{2 \log \kappa} + \left(\frac{1}{2} + \frac{1}{\log \kappa} + \frac{\log a}{\log \kappa}\right) \log x - \left(1 + \frac{\log a}{\log \kappa}\right) \log \log x \\ &+ P_2\left(\log_\kappa \frac{x}{\log_\kappa x}\right) + o(1) \end{aligned}$$

for large  $x$ , where  $P_2(t)$  is a bounded 1-periodic function. We see once again the generality of the asymptotic pattern (1.3).

On the other hand, the function

$$\varpi(x) := \exp\left(\frac{(\log(x/\sqrt{q}))^2}{2 \log(1/q)}\right)$$

satisfies the  $q$ -difference equation

$$\varpi(x) = x\varpi(qx)$$

and is a fundamental factor in the asymptotic theory of  $q$ -difference equations; see the two survey papers [3, 19] and the references therein. This equation will also play an important role in our analysis.

From yet another angle, one easily checks that the series

$$M(x) := \sum_{j \geq 0} \frac{q^{\binom{j}{2}}}{j!} x^j$$

satisfies (1.7). The largest term occurs, by simple calculus, at

$$j \approx \log_{\kappa} x - \log_{\kappa} \log_{\kappa} x + \frac{1}{2} + o(1),$$

and, by the analytic approach we use in this paper, we can deduce that the logarithm of the series is, up to an error of  $O(1)$ , of the same asymptotic order as  $\log \nu(x)$ ; see (1.6) and section 6. The function  $M(x)$  arises sporadically in many different contexts and plays an important role in the corresponding asymptotic estimates; see section 6 for a list of some representative references.

A closely related sum arises in the average-case analysis of a simple backtracking algorithm (see [68]), which corresponds to the expected number of independent sets in a random graph (or, equivalently, the expected number of cliques by interchanging  $q$  and  $p$ )

$$(1.8) \quad J_n := \sum_{1 \leq j \leq n} \binom{n}{j} q^{j(j-1)/2};$$

see [51, 68]. Wilf [68] showed that  $J_n = O(n^{\log n})$  when  $p = 1/2$ . While such a crude bound is easily obtained, the more precise asymptotics of  $J_n$  is nontrivial. First, it is straightforward to check that  $J_n \sim M(n)$  for large  $n$ . Second, the approach we develop in this paper can be used to show that  $J_n$  has an asymptotic expansion similar to (1.3). Indeed, it is readily checked that  $J_n + 1$  satisfies the same recurrence relation as  $\mu_n$  with different initial conditions. So the asymptotics of  $J_n$  follows the same pattern (1.3) as that of  $\mu_n$ ; see section 6 for more details.

Thus examining all independent sets one after another in the backtracking style of Wilf [68] and identifying the one with the maximum cardinality also leads to an expected  $n^{c \log n}$ -complexity.

The diverse aspects we discussed of algorithms or equations leading to the scale  $n^{c \log n}$  are summarized in Figure 1. The bridge connecting the algorithms and the analysis is the binomial recurrence (1.2) as explained above.

This paper is organized as follows. We derive in the next section an asymptotic expansion for  $\mu_n$  using a purely analytic approach. The interest of deriving such a precise asymptotic approximation is at least fourfold.

*Asymptotics:* It goes much beyond the crude description  $n^{c \log n}$  and provides a more precise description; see particularly (1.4) and its implication mentioned there. Indeed, few papers in the literature address such an aspect; see [15, 16, 17, 21, 44, 55, 59].

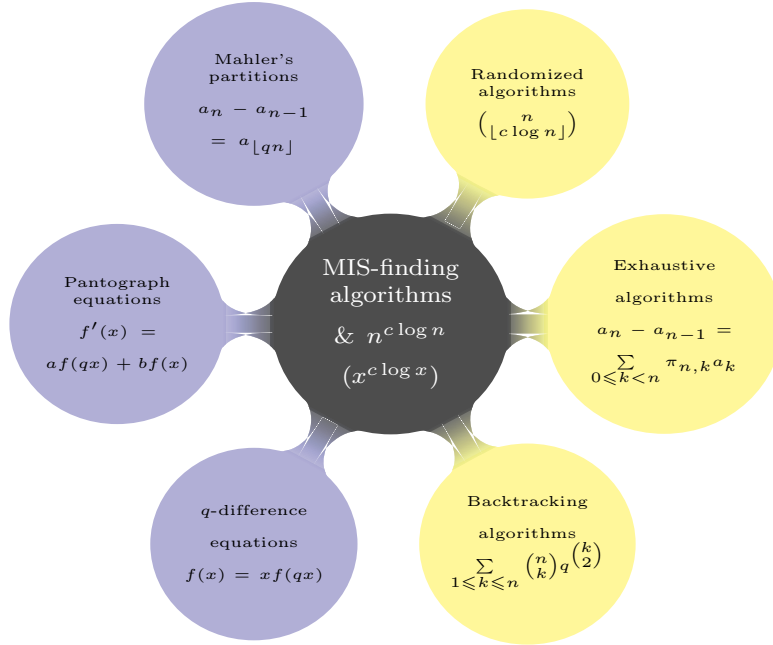


FIG. 1. The connection between MIS-finding algorithms and the scale  $n^{c \log n}$  (discrete) or  $x^{c \log x}$  (continuous). The circles on the right-hand side are more algorithmic in nature, while those on the left-hand side more analytic in nature.

*Numerics:* All scales involved in problems of similar nature here are expressed either in log or in log log, making them more subtle to be identified by numerical simulations. The inherent periodic functions and the slow convergence further add to the complications.

*Methodology:* Our approach, different from previous ones that rely on explicit generating functions (GFs) in product forms, is based on the underlying functional equation and is of some generality; it is akin to some extent to Mahler’s analysis in [50] but gives more precise approximations.

*Generality:* The asymptotic pattern (1.3) is also of some generality, an aspect already examined in details in several papers; see, for example, [16, 17, 21, 44]. See also the last section for a list of diverse contexts where the order  $n^{c \log n}$  appears.

Alternative approaches leading to different asymptotic expansions are discussed in section 3. A rough estimate for  $\mu_n$  was derived in [48] (mentioned in [56]) by an elementary approach.

The next curiosity after the expected value is the variance. But due to strong dependence of the subproblems, the variance is quite challenging at this stage. We consider instead an idealized *independent* version of  $X_n$  (the total cost of the exhaustive algorithm implied by (1.1)), namely,

$$(1.9) \quad Y_n \stackrel{d}{=} Y_{n-1} + Y_{n-1-\text{Binom}(n-1;p)}^* \quad (n \geq 2)$$

with  $Y_1 := 1$  and  $Y_0 := 0$ , where “ $\stackrel{d}{=}$ ” stands for equality in distribution,  $Y_n^*$  is an identical copy of  $Y_n$ , and all the random variables on the right-hand side (including the binomial) are independent. The original random variable  $X_n$  satisfies the same distri-

butional recurrence (see (2.1)) but with the two terms ( $X_{n-1}$  and  $X_{n-1-\text{Binom}(n-1;p)}^*$ ) on the right-hand side *dependent*. We expect that  $Y_n$  would provide insight into the possible stochastic behavior of  $X_n$  although we were unable to evaluate their difference. We show, by a method of moments, that  $Y_n$  is *asymptotically normally distributed* in addition to deriving an asymptotic estimate for the variance. Monte Carlo simulations for  $n$  up to a few hundred show that the limiting distribution of  $X_n$  seems likely to be normal, although the ratio between its variance and that of  $Y_n$  grows like a concave function. But the sample size  $n$  is not large enough to provide more convincing conclusions from simulations. This shows again the subtlety of the problem.

From the recurrence (1.9), it is natural to consider the following variant (by changing the underlying binomial to uniform distribution):

$$(1.10) \quad Z_n \stackrel{d}{=} Z_{n-1} + Z_{\text{Uniform}(0,n-1)}^* \quad (n \geq 2)$$

with  $Z_0 = 0$  and  $Z_1 = 1$ , where  $Z_n^*$  is an identical copy of  $Z_n$  and all the random variables on the right-hand side (including the binomial) are independent. In this case, we prove that the mean is asymptotic to  $cn^{-1/4}e^{2\sqrt{n}}$  and the limit law is *no more normal*. Surprisingly, the mean sequence (which is, up to a factor of  $n!$ , <http://oeis.org/A005189> in the Encyclopedia of Integer Sequences) also occurs in the study of the theory of measurement (and two-sided generalized Fibonacci sequences); see [22, 23].

We conclude this paper with a few remarks and a list of many instances where  $n^{c \log n}$  arises, further clarifications and connections being given elsewhere.

*Notation.* Throughout this paper,  $0 < p < 1$ ,  $q = 1 - p$ , and  $\kappa = 1/q$ .

**2. Expected cost.** We derive asymptotic approximations to  $\mu_n$  in this section by an analytic approach, which is briefly sketched in Figure 2.

**2.1. Preliminaries and main result.** Recall that  $X_n$  denotes the cost used by the exhaustive search algorithm (implied by (1.1)) for finding an MIS in a random graph, and it satisfies the recurrence

$$(2.1) \quad X_n \stackrel{d}{=} X_{n-1} + X_{n-1-\text{Binom}(n-1;p)}^*$$

with  $X_0 = 0$  and  $X_1 = 1$ , where  $X_n^* \stackrel{d}{=} X_n$ , and the two terms on the right-hand side are *dependent*.

From (2.1), we see that the expected value  $\mu_n$  of  $X_n$  satisfies the recurrence (1.2). Our analytic approach then proceeds along the line depicted in Figure 2. While the approach appears standard (see [25, 39]), the major difference is that instead of Mellin transform, we need Laplace transform since the quantity in question is not polynomially bounded. Another technical novelty is the justification of the analytic de-Poissonization for which we rely strongly on the manipulation of functional equations, differing significantly from previous approaches [39].

*GFs.* Let  $f(z) := \sum_{n \geq 0} \mu_n z^n / n!$  denote the exponential GFs of  $\mu_n$ . Then  $f$  satisfies, by (1.2), the equation

$$f'(z) = 1 + f(z) + e^{pz} f(qz)$$

with  $f(0) = 0$ , or, equivalently, denoting by  $\tilde{f}(z) := e^{-z} f(z)$  the Poisson GF of  $\mu_n$ ,

$$(2.2) \quad \tilde{f}'(z) = \tilde{f}(qz) + e^{-z}$$

with  $\tilde{f}(0) = 0$ .

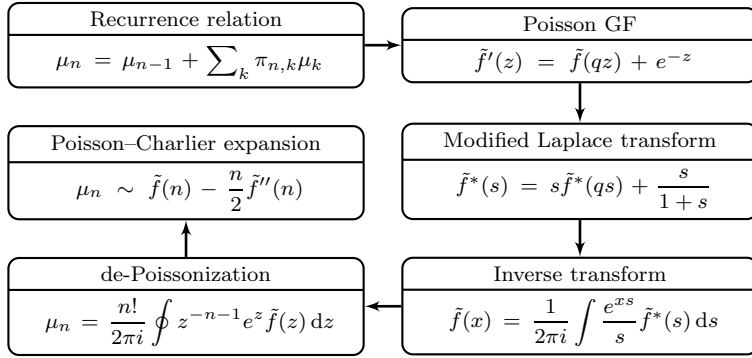


FIG. 2. Our analytic approach to the asymptotics of  $\mu_n$ . Here  $\pi_{n,k} := \binom{n-1}{k} q^k p^{n-1-k}$ .

*Closed-form expressions.* Let  $\tilde{f}(z) = \sum_{n \geq 0} \tilde{\mu}_n z^n / n!$ . From the  $q$ -differential equation (2.2), we derive the recurrence

$$\tilde{\mu}_{n+1} = q^n \tilde{\mu}_n + (-1)^n \quad (n \geq 1).$$

By iteration, we then obtain the closed-form expression

$$\tilde{\mu}_n = \sum_{0 \leq j < n} (-1)^j q^{(n-1-j)(n+j)/2} \quad (n \geq 1).$$

Since  $f(z) = e^z \tilde{f}(z)$ , we then have

$$(2.3) \quad \mu_n = \sum_{1 \leq k \leq n} \binom{n}{k} \sum_{0 \leq j < k} (-1)^j q^{(k-1-j)(k+j)/2} \quad (n \geq 1).$$

This expression is, although exact, less useful for large  $n$ ; also, its asymptotic behavior remains opaque, notably due to the appearance of  $(-1)^j$ . A better expression is given in (3.4), in which all terms are positive.

*Asymptotic approximations.* Our aim in this section is to derive the following asymptotic approximation to  $\mu_n$ .

**THEOREM 2.1.** *The expected cost  $\mu_n$  of the exhaustive search on a random graph satisfies*

$$(2.4) \quad \mu_n = \frac{G\left(\log_\kappa \frac{n}{\log_\kappa n}\right)}{\sqrt{2\pi}} \cdot \frac{n^{1/\log_\kappa n + 1/2}}{\log_\kappa n} \exp\left(\frac{\left(\log \frac{n}{\log_\kappa n}\right)^2}{2 \log \kappa}\right) \left(1 + O\left(\frac{(\log \log n)^2}{\log n}\right)\right),$$

as  $n \rightarrow \infty$ , where  $G(u)$  is defined by ( $\{u\}$  being the fractional part of  $u$ )

$$G(u) = q^{\{\{u\}^2 - \{u\}\}/2} \sum_{j \in \mathbb{Z}} \frac{q^{j(j+1)/2}}{1 + q^{j-\{u\}}} q^{-j\{u\}}$$

(see (2.8)) and is a bounded, 1-periodic function of  $u$ . Note that (2.4) implies (1.3) with

$$P_0(u) = -\frac{1}{2} \log 2\pi - \log \kappa + \log G(u).$$



Our approach leads indeed to an asymptotic expansion, but we content ourselves with the statement of (2.4); see (2.18), (2.23), and (3.3).

The function  $f$  (and thus  $\tilde{f}$ ) is an entire function. It follows immediately that we have the identity

$$\mu_n = \sum_{j \geq 0} \frac{\tilde{f}^{(j)}(n)}{j!} \tau_j(n),$$

(referred to as the Poisson–Charlier expansion in [36]) where the  $\tau_j(n)$ ’s are polynomials of  $n$  of degree  $\lfloor j/2 \rfloor$ ; see (2.24). See also [39] for different representations. However, the hard part always lies in justifying the *asymptotic nature* of the expansion, namely,

$$\mu_n = \sum_{0 \leq j < k} \frac{\tilde{f}^{(j)}(n)}{j!} \tau_j(n) + O\left(n^{\lfloor k/2 \rfloor} \tilde{f}^{(k)}(n)\right)$$

for  $k = 2, 3, \dots$ . In particular, the first-order asymptotic equivalent “ $\mu_n \sim \tilde{f}(n)$ ” is often called the *Poisson heuristic*. Thus the asymptotics of  $\mu_n$  is reduced to that of  $\tilde{f}(x)$  once we justify the asymptotic nature of the expansion. Of special mention is that, unlike almost all papers in the literature, we need only the asymptotic behavior of  $\tilde{f}(x)$  for *real values of  $x$* , all analysis involving complex parameters being carefully handled by the corresponding functional equation.

We will derive an asymptotic expansion for  $\tilde{f}(x)$  for large real  $x$  by Laplace transform techniques and suitable manipulation of the saddle-point method, and then bridge the asymptotics of  $\mu_n$  and  $\tilde{f}(n)$  by a variant of the saddle-point method (or de-Poissonization procedure; see [39]); see Figure 2 for a sketch of our proof.

**2.2. Asymptotics of  $\tilde{f}(x)$ .** We derive an asymptotic expansion for  $\tilde{f}(x)$  in this subsection.

*Modified Laplace transform.* For technical convenience, consider the modified Laplace transform

$$\tilde{f}^*(s) := \frac{1}{s} \int_0^\infty e^{-x/s} \tilde{f}(x) dx.$$

Note that this use of the Laplace transform differs from the usual one by a factor  $1/s$  and by a change of variables  $s \mapsto 1/s$ . Also the use of the exponential GF coupling with this Laplace transform is equivalent to considering the ordinary GF of  $\mu_n$ ; see section 3.2 for more information.

Then the functional-differential equation (2.2) translates into the following functional equation for  $\tilde{f}^*$ :

$$(2.5) \quad \tilde{f}^*(s) = s\tilde{f}^*(qs) + \frac{s}{1+s}$$

for  $\Re(s) > 0$ .

Iterating (2.5) indefinitely, we get

$$(2.6) \quad \tilde{f}^*(s) = \sum_{j \geq 0} \frac{q^{j(j+1)/2}}{1+q^j s} s^{j+1}.$$

We will approximate  $\tilde{f}^*(s)$  for large  $s$  by means of the function

$$F(s) = \sum_{-\infty < j < \infty} \frac{q^{j(j+1)/2}}{1+q^j s} s^{j+1},$$

because adding terms of the form  $s^{-j}$ ,  $j \geq 0$ , does not alter the dominant asymptotic order of both functions for large  $|s|$ .

LEMMA 2.2. For  $x > 1$ , we have

$$(2.7) \quad F(x) = x^{1/2} \exp\left(\frac{(\log x)^2}{2 \log \kappa}\right) G(\log_\kappa x),$$

where

$$(2.8) \quad G(u) := q^{\{u\}^2 + \{u\}} F\left(q^{-\{u\}}\right)$$

is a continuous, positive, periodic function with period 1.

*Proof.* One can easily check that  $F(s)$  satisfies a functional equation similar to that of Jacobi's theta functions

$$(2.9) \quad F(s) = sF(qs) \quad (s \in \mathbb{C}).$$

Iterating  $N$  times this functional equation, we obtain

$$(2.10) \quad F(s) = q^{N(N-1)/2} s^N F(q^N s) \quad (s \in \mathbb{C}).$$

Assume  $x > 1$ . Take

$$N = \lfloor \log_\kappa x \rfloor = \log_\kappa x + \eta,$$

where  $\eta = -\{\log_\kappa x\}$ . Then we have

$$\begin{aligned} F(x) &= \exp\left(\frac{N(N-1)}{2} \log q + N \log x\right) F(e^{N \log q + \log x}) \\ &= \exp\left(\frac{(\log x)^2}{2 \log \kappa} + \frac{\log x}{2} + \frac{\eta(\eta-1)}{2} \log q\right) F(e^{\eta \log q}) \\ &= q^{(\eta^2 - \eta)/2} x^{1/2} \exp\left(\frac{(\log x)^2}{2 \log \kappa}\right) F(e^{\eta \log q}), \end{aligned}$$

which, together with the functional equation  $F(1/q) = F(1)/q$  (or  $G(u+1) = G(u)$ ), proves the lemma.  $\square$

*Asymptotic expansion of  $\tilde{f}(x)$ : Saddle-point method.* By the Laplace inversion formula with suitable change of variables, we have

$$(2.11) \quad \tilde{f}(x) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} \frac{e^{xs}}{s} \tilde{f}^* \left(\frac{1}{s}\right) ds,$$

where  $r > 0$  is a small number whose value will be specified later. We now derive a few estimates for  $\tilde{f}^*(s)$ .

LEMMA 2.3. (i) If  $r > 0$  and  $|t| \geq 1$ , then

$$(2.12) \quad \tilde{f}^* \left(\frac{1}{r+it}\right) = O\left(\frac{1}{|t|}\right);$$

(ii) if  $0 < r \leq 1$  and  $|t| \leq 1$ , then

$$(2.13) \quad \tilde{f}^* \left(\frac{1}{r+it}\right) = F\left(\frac{1}{r+it}\right) + O(1);$$

(iii) if  $r > 0$  and  $c_m r \leq |t| \leq 1$ , where  $c_m := \sqrt{q^{-2m} - 1}$ ,  $m \geq 1$ , then

$$(2.14) \quad \tilde{f}^* \left( \frac{1}{r + it} \right) = O \left( r^m q^{\binom{m}{2}} F \left( \frac{1}{r} \right) \right).$$

*Proof.* First, (2.12) follows from (2.6) since  $\tilde{f}^*(s) = O(|s|)$  as  $|s| \rightarrow 0$ . For the estimate (2.13), we observe that

$$\left| \frac{1}{1 + sq^j} \right| \leq \min\{q^{-j}|s|^{-1}, 1\} \quad (\Re(s) \geq 0).$$

Then

$$\tilde{f}^*(s) = F(s) + O(|s|^{-1})$$

for  $\Re(s) \geq 0$  and  $|s| \geq c > 0$ . Also, for  $r > 0$ ,

$$\Re \left( \frac{1}{r + it} \right) = \frac{r}{r^2 + t^2} > 0;$$

and, for  $|t| \leq 1$  and  $0 < r \leq 1$ ,

$$\frac{1}{|r + it|} \geq \frac{1}{\sqrt{2}}.$$

From these two estimates, we then deduce (2.13).

On the other hand, if  $\Re(s) \geq 0$ , then

$$|\tilde{f}^*(s)| \leq \sum_{j \geq 0} q^{j(j+1)/2} |s|^{j+1} \leq \vartheta(|s|),$$

where

$$\vartheta(x) := \sum_{-\infty < j < \infty} q^{j(j-1)/2} x^j.$$

It is easily checked that  $\vartheta(x)$  satisfies the same functional equation (2.9) as  $F(x)$ , namely,

$$\vartheta(x) = x\vartheta(qx).$$

Thus, by the same arguments used for proving (2.7), we have, for  $x > 1$ ,

$$\vartheta(x) = x^{1/2} \exp \left( \frac{(\log x)^2}{2 \log \kappa} \right) g(\log_\kappa x),$$

where  $g(x)$  is a continuous, bounded, periodic function. Comparing this expression with (2.7) for  $F(x)$ , we conclude that  $\vartheta(x) = O(F(x))$  for  $x \geq 1$ .

Let  $c_m := \sqrt{q^{-2m} - 1}$ ,  $m > 1$ . Then, for  $0 < r < 1$ ,

$$\begin{aligned} \max_{c_m r \leq |t| \leq 1} \left| \tilde{f}^* \left( \frac{1}{r + it} \right) \right| &\leq \max_{c_m r \leq |t| \leq 1} \left| \vartheta \left( \frac{1}{\sqrt{r^2 + t^2}} \right) \right| \\ &= \vartheta(q^m/r) \\ &= r^m q^{m(m-1)/2} \vartheta(1/r) \\ &= O \left( r^m q^{\binom{m}{2}} F(1/r) \right). \end{aligned}$$

This proves (2.14) and the lemma.  $\square$

By splitting the integral in (2.11) into three ranges  $|t| \leq c_m r$ ,  $c_m r < |t| \leq 1$ , and  $|t| > 1$ , where  $t = \Im(s)$ , and then applying the estimates (2.12) and (2.14), we deduce that

$$(2.15) \quad \tilde{f}(x)e^{-xr} = I_r(x) + O\left(r^{m-1}q^{\binom{m}{2}}F(1/r) + 1\right),$$

where

$$I_r(x) := \frac{1}{2\pi} \int_{-c_m r}^{c_m r} \frac{e^{ixt}}{r + it} F\left(\frac{1}{r + it}\right) dt.$$

It remains to evaluate more precisely the integral  $I_r(x)$  by the saddle-point method.

We now take

$$N = \lfloor \log_\kappa(1/r) \rfloor = \log_\kappa(1/r) + \eta,$$

where  $\eta = -\{\log_\kappa(1/r)\}$ . Applying the functional equation (2.10) with  $s = 1/(r + it)$ , we get

$$I_r(x) = \frac{1}{2\pi} \int_{-c_m r}^{c_m r} \frac{e^{ixt} q^{N(N-1)/2}}{(r + it)^{N+1}} F\left(\frac{rq^\eta}{r + it}\right) dt.$$

By the relation

$$F(1/r) = q^{N(N-1)/2} r^{-N} F(q^\eta),$$

we then have

$$\begin{aligned} I_r(x) &= \frac{F(1/r)}{2\pi r} \int_{-c_m r}^{c_m r} e^{ixt} \left(\frac{r}{r + it}\right)^{N+1} \frac{F(rq^\eta/(r + it))}{F(q^\eta)} dt \\ &= \frac{F(1/r)e^{xr}}{2\pi} \int_{-c_m}^{c_m} e^{ixrt} \left(\frac{1}{1 + it}\right)^{N+1} \frac{F(q^\eta/(1 + it))}{F(q^\eta)} dt \\ &= \frac{F(1/r)e^{xr}}{2\pi} \int_{-c_m}^{c_m} e^{-xrt^2/2} H(t) dt, \end{aligned}$$

where

$$H(t) := e^{xr(it - \log(1+it) + t^2/2)} \frac{F(q^\eta/(1 + it))}{(1 + it)^{1+\eta} F(q^\eta)}.$$

So far the choice of  $r > 0$  is arbitrary. For an optimal choice, take  $r = r(x) > 0$  to be the approximate saddle-point such that

$$(2.16) \quad \frac{1}{r} \log \frac{1}{r} = x \log \kappa.$$

Note that  $r$  can be expressed in terms of the Lambert-W function (principal solution of the equation  $W(x)e^{W(x)} = x$ ) as

$$r = \frac{W(x \log \kappa)}{x \log \kappa};$$

thus  $\log(1/r) = W(x \log \kappa)$ . Asymptotically,

(2.17)

$$W(x) = \log x - \log \log x + \frac{\log \log x}{\log x} + \frac{(\log \log x)^2 - 2 \log \log x}{2(\log x)^2} + O\left(\frac{(\log \log x)^3}{(\log x)^3}\right),$$

as  $x \rightarrow \infty$ ; see [14] for more information on  $W$ .

Since  $m > 1$  is arbitrary and  $r \asymp x^{-1} \log x$ , the relation (2.15) is an asymptotic approximation, albeit less explicit.

To derive a more explicit expansion, we first observe that

$$e^{xr} F(1/r) = r^{-1/\log \kappa - 1/2} e^{(\log(1/r))^2/(2 \log \kappa)} G(\log_\kappa(1/r)),$$

by (2.7) and (2.16). Then what remains is standard (see [25]): we evaluate the integral in (2.15) by Laplace’s method (a change of variable  $t \mapsto t/\sqrt{xr}$  followed by an asymptotic expansion of  $H(t/\sqrt{xr})$  for large  $xr$  and then an integration term by term), and we obtain the following expansion.

PROPOSITION 2.4. *With  $r$  given by (2.16),  $\tilde{f}(x)$  satisfies the asymptotic expansion* (2.18)

$$\tilde{f}(x) \sim \frac{e^{(\log(1/r))^2/(2 \log \kappa)} G(\log_\kappa(1/r))}{r^{1/\log \kappa + 1/2} \sqrt{2\pi} \log_\kappa(1/r)} \left( 1 + \sum_{j \geq 1} \phi_j(\log_\kappa(1/r)) (\log_\kappa(1/r))^{-j} \right),$$

as  $x \rightarrow \infty$ , where  $G$  is given in (2.8) and the  $\phi_j(u)$ ’s are bounded, 1-periodic functions of  $u$  involving the derivatives of  $F(q^{-\{u\}})$ .

In particular,

$$\phi_1(u) = - \left( \frac{1}{12} - \frac{\{u\}(1 - \{u\})}{2} + \frac{(1 - \{u\})q^{-\{u\}}F'(q^{-\{u\}})}{F(q^{-\{u\}})} + \frac{q^{-2\{u\}}F''(q^{-\{u\}})}{2F(q^{-\{u\}})} \right).$$

By using (2.17), the leading term in (2.18) can be expressed completely in terms of  $\log x$  as follows.

COROLLARY 2.5. *As  $x \rightarrow \infty$ ,  $\tilde{f}(x)$  satisfies*

(2.19)

$$\tilde{f}(x) = \frac{G\left(\log_\kappa \frac{x}{\log_\kappa x}\right)}{\sqrt{2\pi}} \cdot \frac{x^{1/\log \kappa + 1/2}}{\log_\kappa x} \exp\left(\frac{\left(\log \frac{x}{\log_\kappa x}\right)^2}{2 \log \kappa}\right) \left(1 + O\left(\frac{(\log \log x)^2}{\log x}\right)\right).$$

This gives Theorem 2.1 with  $x$  here replaced by  $n$ .

As another consequence, we see, by (2.2) and (2.19), that

$$\frac{\tilde{f}'(x)}{\tilde{f}(x)} \sim \frac{\tilde{f}'(qx)}{\tilde{f}(qx)} \sim \frac{\log_\kappa x}{x}.$$

More generally, we have the following asymptotic relations for  $\tilde{f}^{(j)}(x)$  and  $\tilde{f}(q^j x)$ .

COROLLARY 2.6. *For  $j \geq 1$ ,*

(2.20) 
$$\frac{\tilde{f}^{(j)}(x)}{\tilde{f}(x)} \sim \left(\frac{\log_\kappa x}{x}\right)^j,$$

(2.21) 
$$\frac{\tilde{f}(q^j x)}{\tilde{f}(x)} \sim q^{-j(j-1)/2} \left(\frac{\log_\kappa x}{x}\right)^j.$$

Note that (2.20) also follows easily from the integral representation

$$\tilde{f}^{(j)}(x) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} \frac{e^{xs}}{s^{j-1}} \tilde{f}^* \left( \frac{1}{s} \right) ds$$

and exactly the same arguments used above.

**2.3. Asymptotics of  $\mu_n$ .** We first derive a simple lemma for the ratio  $f(x + y)/f(x)$  when  $y$  is not too large by using (2.20).

LEMMA 2.7. *Assume  $x > 1$ . If  $|y| = o(x/\log x)$ , then*

$$(2.22) \quad \frac{\tilde{f}(x + y)}{\tilde{f}(x)} = 1 + O\left(\frac{|y| \log x}{x}\right).$$

*Proof.* By (2.20), we have

$$\begin{aligned} \log \frac{\tilde{f}(x + y)}{\tilde{f}(x)} &= y \int_0^1 \frac{\tilde{f}'(x + yt)}{\tilde{f}(x + yt)} dt \\ &= yO\left(\int_0^1 \frac{\log|x + yt|}{|x + yt|} dt\right) \\ &= O\left(\frac{|y| \log|x|}{|x|}\right), \end{aligned}$$

from which (2.22) follows.  $\square$

THEOREM 2.8. *The expected cost used by the exhaustive search algorithm satisfies the asymptotic expansion*

$$(2.23) \quad \mu_n \sim \tilde{f}(n) + \sum_{j \geq 2} \frac{\tilde{f}^{(j)}(n)}{j!} \tau_j(n),$$

where  $\tau_j(n)$  is a (Charlier) polynomial in  $n$  of degree  $\lfloor j/2 \rfloor$  defined by

$$(2.24) \quad \tau_j(n) := \sum_{0 \leq \ell \leq j} \binom{j}{\ell} (-1)^\ell \frac{n! n^\ell}{(n - j + \ell)!} \quad (j = 0, 1, \dots).$$

In particular,  $\tau_0(n) = 1$ ,  $\tau_1(n) = 0$ ,  $\tau_2(n) = -n$ ,  $\tau_3(n) = 2n$ , and  $\tau_4(n) = 3n^2 - 6n$ . Thus, by (2.18) and (2.20),

$$\mu_n = \tilde{f}(n) \left(1 + O(n^{-1}(\log n)^2)\right),$$

which proves Theorem 2.1.

*Proof.* For simplicity, we prove only the following estimate:

$$(2.25) \quad \mu_n = \tilde{f}(n) - \frac{n}{2} \tilde{f}''(n) + O\left(n^{-2}(\log n)^4 \tilde{f}(n)\right).$$

The same method of proof easily extends to the proof of (2.23).

We start with the Taylor expansion of  $\tilde{f}(z)$  at  $z = n$  to the fourth order

$$(2.26) \quad \tilde{f}(z) = \sum_{0 \leq j \leq 3} \frac{\tilde{f}^{(j)}(n)}{j!} (z - n)^j + (z - n)^4 R(z),$$

where

$$R(z) = \frac{1}{3!} \int_0^1 \tilde{f}^{(4)}(n + (z - n)t)(1 - t)^3 dt.$$

By applying successively (2.2), we get

$$\tilde{f}^{(4)}(z) = -e^{-z} + q^3 e^{-qz} - q^5 e^{-q^2 z} + q^6 e^{-q^3 z} + q^6 \tilde{f}(q^4 z).$$

It follows that

$$\begin{aligned} |R(ne^{i\theta})| &\leq \int_0^1 |\tilde{f}^{(4)}(n + n(e^{i\theta} - 1)t)| dt \\ &= O\left(e^{-n \cos \theta} + e^{-q^3 n \cos \theta} + \int_0^1 |\tilde{f}(q^4 n + q^4 n(e^{i\theta} - 1)t)| dt\right) \end{aligned}$$

for  $|\theta| \leq \pi$ . Replacing first  $\tilde{f}(z)$  inside the integral by  $e^{-z} f(z)$ , using the inequality  $|f(z)| \leq f(|z|)$ , and then substituting back  $f(q^4 n)$  by  $e^{q^4 n} \tilde{f}(q^4 n)$ , we then have

$$\begin{aligned} |R(ne^{i\theta})| &= O\left(e^{-q^3 n \cos \theta} + f(q^4 n) \int_0^1 |e^{-q^4 n - q^4 n(e^{i\theta} - 1)t}| dt\right) \\ &= O\left(e^{-q^3 n \cos \theta} + \tilde{f}(q^4 n) \int_0^1 e^{q^4 n(1 - \cos \theta)t} dt\right) \\ (2.27) \quad &= O\left(e^{-q^3 n \cos \theta} + \tilde{f}(q^4 n) e^{q^4 n(1 - \cos \theta)}\right), \end{aligned}$$

uniformly for  $|\theta| \leq \pi$ . By Cauchy’s integral formula and (2.26), we have

$$\begin{aligned} \mu_n &= \frac{n!}{2\pi i} \oint_{|z|=n} z^{-n-1} e^z \tilde{f}(z) dz \\ &= \frac{n!}{2\pi i} \oint_{|z|=n} z^{-n-1} e^z \left( \sum_{0 \leq j \leq 3} \frac{\tilde{f}^{(j)}(n)}{j!} (z - n)^j \right) dz + R_n \\ &= \tilde{f}(n) - \frac{n}{2} \tilde{f}''(n) + \frac{n}{3} \tilde{f}'''(n) + R_n, \end{aligned}$$

where

$$R_n := \frac{n!}{2\pi i} \oint_{|z|=n} z^{-n-1} e^z (z - n)^4 R(z) dz.$$

By the estimate (2.27) for  $R(z)$ , we have

$$\begin{aligned} R_n &= O\left(n! n^{4-n} \int_{-\pi}^{\pi} \theta^4 e^{n \cos \theta} |R(ne^{i\theta})| d\theta\right) \\ &= O\left(n! n^{4-n} \int_{-\pi}^{\pi} \theta^4 e^{n \cos \theta} \left(e^{-q^3 n \cos \theta} + \tilde{f}(q^4 n) e^{q^4 n(1 - \cos \theta)}\right) d\theta\right) \\ &= O\left(n! n^{4-n} \int_{-\pi}^{\pi} \theta^4 e^{n(1 - q^3) \cos \theta} d\theta + n! \tilde{f}(q^4 n) n^{4-n} e^n \int_{-\pi}^{\pi} \theta^4 e^{-(1 - q^4)n(1 - \cos \theta)} d\theta\right) \\ &= O\left(n! n^{-n+3/2} e^{(1 - q^3)n} + n! e^n n^{-n+3/2} \tilde{f}(q^4 n)\right) \\ &= O\left(n^2 e^{-q^3 n} + n^2 \tilde{f}(q^4 n)\right) \\ &= O\left(n^{-2} (\log n)^4 \tilde{f}(n)\right), \end{aligned}$$

by (2.21). Note that again by (2.20),

$$n\tilde{f}'''(n) = O\left(n^{-2}(\log n)^3\tilde{f}(n)\right),$$

so this error bound is absorbed in  $O(\tilde{f}(n)n^{-2}(\log n)^4)$ . This proves (2.25).  $\square$

**3. Alternative expansions and approaches.** For more methodological interest, we discuss in this section other possible approaches to the asymptotic expansions we derived above.

**3.1. An alternative expansion for  $\tilde{f}(x)$ .** We begin with an alternative asymptotic expansion for  $\tilde{f}(x)$ , starting from the integral representation (2.11), which, as showed above, can be approximated by

$$\tilde{f}(x) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} \frac{e^{xs}}{s} F\left(\frac{1}{s}\right) ds + O(1).$$

For simplicity, we drop the  $O(1)$ -term and write this as

$$\tilde{f}(x) \simeq \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} \frac{e^{xs}}{s} F\left(\frac{1}{s}\right) ds.$$

Now we use the same  $N = \lfloor \log_\kappa(1/r) \rfloor = \log_\kappa(1/r) - \eta$  and

$$F\left(\frac{1}{s}\right) = q^{N(N-1)/2} s^{-N} F\left(\frac{q^N}{s}\right),$$

so that

$$(3.1) \quad \tilde{f}(x) \simeq \frac{q^{\binom{N}{2}}}{2\pi i} \int_{r-i\infty}^{r+i\infty} \frac{e^{xs}}{s^{N+1}} F\left(\frac{q^N}{s}\right) ds.$$

Now instead of expanding  $F(q^N/(r+it))$  at  $t = 0$ , we expand  $F(q^N/s)$  at  $s = r$ , giving

$$F\left(\frac{q^N}{s}\right) = F\left(\frac{q^N}{r} - \frac{q^N}{r} \left(1 - \frac{r}{s}\right)\right) = \sum_{m \geq 0} \frac{(-1)^m Q^m}{m!} F_m \left(1 - \frac{r}{s}\right)^m,$$

where  $Q := q^N/r = q^{-\{\log_\kappa(1/r)\}}$  and  $F_j$  denotes  $F^{(j)}(Q)$ . Substituting this expansion into the integral representation (3.1) and then integrating term-by-term, we obtain

$$(3.2) \quad \begin{aligned} \tilde{f}(x)q^{-\binom{N}{2}} &\simeq \sum_{m \geq 0} \frac{(-1)^m Q^m}{m!} F_m \cdot \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} \frac{e^{xs}}{s^{N+1}} \left(1 - \frac{r}{s}\right)^m ds \\ &= \frac{x^N}{N!} \sum_{m \geq 0} \frac{(-1)^m Q^m}{m!} F_m T_m(N), \end{aligned}$$

where, by the integral representation for Gamma function (see [25]),

$$\begin{aligned} T_m(N) &:= \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} \frac{e^{xs}}{s^{N+1}} \left(1 - \frac{r}{s}\right)^m ds \\ &= \sum_{0 \leq j \leq m} \binom{m}{j} (-r)^j \frac{N! x^j}{(N+j)!}. \end{aligned}$$



For computational purposes, it is preferable to use the recurrence

$$T_m(N) = T_{m-1}(N) - \frac{rx}{N+1} T_{m-1}(N+1).$$

The value of  $r$  is arbitrary up to now. If we take  $r = N/x$ , then

$$T_m(N) := \sum_{0 \leq j \leq m} \binom{m}{j} (-1)^j \frac{N! N^j}{(N+j)!}.$$

Note that  $|T_m(N)| \asymp N^{-\lceil m/2 \rceil}$ . In particular,

$$T_0(N) = 1, T_1(N) = \frac{1}{N+1}, T_3(N) = -\frac{N-2}{(N+1)(N+2)}, \dots$$

Since  $q^N/r$  remains bounded, we can regroup the terms and get an asymptotic expansion in terms of increasing powers of  $N^{-1}$ , the first few terms being given as follows:

$$\frac{\tilde{f}(x)}{q^{\binom{N}{2}} x^N / N!} \simeq F_0 - \frac{Q(2F_1 + F_2Q)}{2N} + \frac{Q(3F_4Q^3 + 28F_3Q^2 + 60F_2Q + 24F_1)}{24N^2} + \dots$$

On the other hand, if we choose  $r = (N+1)/x$ , then  $T_1(N) = 0$  and

$$T_0(N) = 1, T_2(N) = -\frac{1}{N+2}, T_3(N) = -\frac{4}{(N+2)(N+3)}, \dots,$$

so that

$$\frac{\tilde{f}(x)}{q^{\binom{N}{2}} x^N / N!} \simeq F_0 - \frac{F_2Q^2}{2(N+2)} + \frac{Q^3(3F_4Q + 16F_3)}{24(N+2)^2} + \dots$$

While  $|T_m(N)| \asymp N^{-\lceil m/2 \rceil}$  for  $m \geq 2$  as in the case of  $r = N/x$ , this is a better expansion because the first term incorporates more information.

The more transparent expansion (3.2) is a priori a *formal* one whose asymptotic nature can be justified by the same local analysis as above. We summarize the analysis in the following theorem.

**THEOREM 3.1.** *The Poisson GF of  $\mu_n$  satisfies the asymptotic expansion*

$$(3.3) \quad \tilde{f}(x) \sim q^{\binom{N}{2}} \frac{x^N}{N!} \sum_{m \geq 0} \frac{(-1)^m Q^m}{m!} F^{(m)}(Q) T_m(N),$$

where  $N = \lfloor \log_\kappa(1/r) \rfloor = \log_\kappa(1/r) - \eta$ ,  $r := N/x$ ,  $Q := q^{-\log_\kappa(1/r)}$ , and  $T_m(N)$  is defined by

$$T_m(N) := \sum_{0 \leq j \leq m} \binom{m}{j} (-1)^j \frac{N!(N+1)^j}{(N+j)!}.$$

Straightforward calculations give (when  $r = N/x$ )

$$\begin{aligned} \log \left( q^{\binom{N}{2}} \frac{x^N}{N!} \right) &= \frac{\left( \log \frac{x}{\log_\kappa x} \right)^2}{2 \log \kappa} + \left( \frac{1}{\log \kappa} + \frac{1}{2} \right) \log x - \log \log x \\ &\quad - \frac{1}{2} \log 2\pi - \frac{\eta^2 + \eta}{2} + O \left( \frac{(\log \log x)^2}{\log x} \right), \end{aligned}$$

consistent with what we proved in (2.19) via directly applying the saddle-point method. For similar types of approximation, see [33, 50].

**3.2. Exponential GFs vs. ordinary GFs.** The different forms of the GFs of the sequence  $\mu_n$  have several interesting features which we now briefly explore.

Instead of  $\tilde{f}^*(s)$ , we start with considering the usual Laplace transform of  $\tilde{f}(z)$

$$\mathcal{L}(s) = \int_0^\infty e^{-xs} \tilde{f}(x) dx,$$

which, by (2.6), satisfies

$$\mathcal{L}(s) = \sum_{j \geq 0} \frac{q^{\binom{j+1}{2}}}{s^{j+1}(s+q^j)}.$$

By inverting this series, we obtain

$$\tilde{f}(z) = \sum_{j \geq 0} \frac{q^{\binom{j+1}{2}}}{j!} z^{j+1} \int_0^1 e^{-q^j uz} (1-u)^j du.$$

From this exact expression, we deduce not only the exact expression (2.3) but also the following one (by multiplying both sides by  $e^z$  and then expanding):

$$(3.4) \quad \mu_n = n \sum_{0 \leq j < n} \binom{n-1}{j} q^{\binom{j+1}{2}} \sum_{0 \leq \ell < n-j} \binom{n-1-j}{\ell} \frac{q^{j\ell} (1-q^j)^{n-1-j-\ell}}{j+\ell+1},$$

where all terms are now positive; compare (2.3). This expression is more useful than (2.3) for numerical purposes but still requires an  $O(n^2)$  complexity. For large  $n$ , it is preferable to use the asymptotic expansions derived above.

On the other hand, the consideration of our  $\tilde{f}^*(s)$  bridges essentially exponential GF (EGF) and ordinary GF (OGF) of  $\mu_n$ . Indeed,

$$\begin{aligned} \tilde{f}^*(s) &= \frac{1}{s} \int_0^\infty e^{-x-x/s} \sum_{n \geq 0} \frac{\mu_n}{n!} x^n dx \\ &= \frac{1}{1+s} \sum_{n \geq 0} \mu_n \left( \frac{s}{1+s} \right)^n, \end{aligned}$$

which is essentially the Euler transform of the OGF; see [24].

Our proofs given above rely strongly on the use of EGF, but the use of OGF works equally well. We consider the general recurrence

$$(3.5) \quad a_n = a_{n-1} + \sum_{0 \leq j < n} \pi_{n,j} a_j + b_n \quad (n \geq 1)$$

with  $a_0$  and  $\{b_n\}_{n \geq 1}$  given. Then the OGF  $A(z) := \sum_{n \geq 1} a_n z^n$  satisfies

$$A(z) = zA(z) + \frac{z}{1-pz} A\left(\frac{qz}{1-pz}\right) + B(z),$$

where  $B(z) := \sum_{n \geq 1} b_n z^n$ . Thus  $\bar{A}(z) := (1-z)A(z)$  satisfies

$$\bar{A}(z) = B(z) + \frac{z}{1-z} \bar{A}\left(\frac{qz}{1-pz}\right),$$

which after iteration gives

$$\bar{A}(z) = \sum_{j \geq 0} q^{j(j-1)/2} \left( \frac{z}{1-z} \right)^j B \left( \frac{q^j z}{1 - (1-q^j)z} \right).$$

Thus

$$(3.6) \quad A(z) = \sum_{j \geq 0} \frac{q^{j(j-1)/2} z^j}{(1-z)^{j+1}} B \left( \frac{q^j z}{1 - (1-q^j)z} \right).$$

Closed-form expressions can be derived from this; we omit the details here.

**4. Variance of  $Y_n$ .** We derive in this section the asymptotics of the variance  $Y_n$  (see (1.9)), which can be regarded as a very rough independent approximation to  $X_n$ . We use an elementary approach (no complex analysis being needed) here based on the recurrences of the central moments and suitable tools of “asymptotic transfer” for the underlying recurrence. The approach is, up to the technical development of the required asymptotic transfer tools, by now standard; see [35, 37]. The same analysis provided here is also applicable to higher central moments, which will be analyzed in the next section.

**4.1. Recurrence.** For the variance of  $Y_n$ , we start with the recurrence (1.9), which translates into the recurrence satisfied by the moment GF  $M_n(y) := \mathbb{E}(e^{Y_n y})$

$$M_n(y) = M_{n-1}(y) \sum_{0 \leq j < n} \pi_{n,j} M_j(y) \quad (n \geq 2)$$

with  $M_0(y) = 1$  and  $M_1(y) = e^y$ , where  $\pi_{n,j} := \binom{n-1}{j} q^j p^{n-1-j}$ . This implies, with  $\bar{M}_n(y) := e^{-\mu_n y} M_n(y) = \mathbb{E}(e^{(Y_n - \mu_n)y})$ , that

$$(4.1) \quad \bar{M}_n(y) = \bar{M}_{n-1}(y) \sum_{0 \leq j < n} \pi_{n,j} \bar{M}_j(y) e^{\Delta_{n,j} y} \quad (n \geq 2)$$

with  $\bar{M}_n(y) = 1$  for  $n < 2$ , where

$$\Delta_{n,j} := \mu_j + \mu_{n-1} - \mu_n.$$

Let  $M_{n,m} := \mathbb{E}(Y_n - \mu_n)^m = \bar{M}_n^{(m)}(0)$ ,  $m \geq 0$ . Then from (4.1), we deduce that

$$(4.2) \quad M_{n,m} = M_{n-1,m} + \sum_{0 \leq j < n} \pi_{n,j} M_{j,m} + T_{n,m},$$

where, for  $m \geq 1$ ,

$$(4.3) \quad \begin{aligned} T_{n,m} &= \sum_{\substack{k+\ell+h=m \\ 0 \leq k, \ell < m \\ 0 \leq h \leq m}} \binom{m}{k, \ell, h} M_{n-1,k} \sum_{0 \leq j < n} \pi_{n,j} M_{j,\ell} \Delta_{n,j}^h \\ &= \sum_{0 \leq \ell < m} \binom{m}{\ell} \sum_{0 \leq j < n} \pi_{n,j} M_{j,\ell} \Delta_{n,j}^{m-\ell} \\ &+ \sum_{2 \leq k \leq m-2} \binom{m}{k} M_{n-1,k} \sum_{0 \leq \ell \leq m-k} \binom{m-k}{\ell} \sum_{0 \leq j < n} \pi_{n,j} M_{j,\ell} \Delta_{n,j}^{m-k-\ell}. \end{aligned}$$

Note that since  $M_{n,1} = 0$  and  $\sum_{0 \leq j < n} \pi_{n,j} \Delta_{n,j} = 0$ , the terms with  $k = 1$  and  $k = m - 1$  vanish.

In particular, the variance  $\sigma_n^2 = M_{n,2}$  satisfies

$$\sigma_n^2 = \sigma_{n-1}^2 + \sum_{0 \leq j < n} \pi_{n,j} \sigma_j^2 + T_{n,2},$$

where

$$T_{n,2} = \sum_{0 \leq j < n} \pi_{n,j} \Delta_{n,j}^2.$$

This will be useful for our asymptotic analysis for  $\sigma_n^2$ .

**4.2. Asymptotics of  $T_{n,2}$ .** To proceed further, we first consider the asymptotics of  $\Delta_{n,j}$  for  $j = qn + O(n^{2/3})$ . By Taylor expansion and (2.2), we have

$$\begin{aligned} \tilde{f}(n) - \tilde{f}(n-1) &= \tilde{f}'(n) - \frac{\tilde{f}''(n)}{2} + \frac{\tilde{f}'''(n)}{3!} + O\left(\int_0^1 (1-t)^4 \tilde{f}^{(4)}(n-t) dt\right) \\ &= \tilde{f}'(n) - \frac{\tilde{f}''(n)}{2} + \frac{\tilde{f}'''(n)}{3!} + O\left(\tilde{f}(q^4 n)\right), \end{aligned}$$

and

$$\tilde{f}''(n) - \tilde{f}''(n-1) = \tilde{f}'''(n) + O\left(\tilde{f}(q^4 n)\right).$$

These and (2.25) yield

$$\begin{aligned} \mu_n - \mu_{n-1} &= \tilde{f}'(n) - \frac{\tilde{f}''(n)}{2} + O\left(n^2 \tilde{f}(q^4 n)\right) \\ &= \tilde{f}(qn) + O\left(n^2 \tilde{f}(q^4 n)\right), \end{aligned}$$

since  $\tilde{f}(q^2 n) = O(n^2 (\log n)^{-2} \tilde{f}(q^4 n))$ . Then, for  $j = qn + x\sqrt{pqn}$ ,  $|x| \leq n^{1/6}$ ,

$$\begin{aligned} \Delta_{n,j} &= \mu_j - (\mu_n - \mu_{n-1}) \\ &= \tilde{f}(qn + x\sqrt{pqn}) - \tilde{f}(qn) + O\left(n^2 \tilde{f}(q^4 n)\right) \\ (4.4) \quad &= \tilde{f}'(qn)x\sqrt{pqn} + O\left(n^2(1+x^2)\tilde{f}(q^4 n)\right). \end{aligned}$$

Thus, by (2.20) and (2.21),

$$\begin{aligned} T_{n,2} &= \sum_{|x| \leq n^{1/6}} \pi_{n,j} \left| \tilde{f}'(qn)x\sqrt{pqn} + O(n^2 \tilde{f}(q^4 n)) \right|^2 + O\left(\mu_n^2 \sum_{|x| > n^{1/6}} \pi_{n,j}\right) \\ &= pqn \tilde{f}'(qn)^2 \sum_{|x| \leq n^{1/6}} \pi_{n,j} |x|^2 + O\left(n^{9/2} \tilde{f}^2(q^4 n)\right) \\ &= pqn \tilde{f}'(qn)^2 + O\left(n^{9/2} \tilde{f}^2(q^4 n)\right) \\ (4.5) \quad &\sim q^{-1} p n^{-3} (\log_\kappa n)^4 \tilde{f}(n)^2. \end{aligned}$$

The next step then is to “transfer” this estimate to the asymptotics of the variance.

**4.3. Asymptotic transfer.** We now develop an asymptotic transfer result, which will be used to compute the asymptotics of higher central moments of  $Y_n$  (in particular the variance).

More generally, we consider a sequence  $\{a_n\}_{n \geq 0}$  satisfying the recurrence relation (3.5), where  $a_0$  is finite (whose value is immaterial) and  $\{b_n\}_{n \geq 1}$  is a given sequence.

LEMMA 4.1. *If  $b_n \sim n^\beta (\log n)^\xi \tilde{f}(n)^\alpha$ , where  $\alpha > 0$ ,  $\beta, \xi \in \mathbb{R}$ , then*

$$\sum_{j \leq n} b_j \sim \frac{n}{\alpha \log_\kappa n} b_n.$$

*Proof.* Define  $\varphi(t) := t^\beta (\log t)^\xi \tilde{f}(t)^\alpha$ . By assumption,  $b_n \sim \varphi(n)$ . Since  $\tilde{f}'(t)/\tilde{f}(t) \sim t^{-1} \log_\kappa t$  (by (2.20)), we see that  $\varphi'(t) > 0$  for  $t$  sufficiently large, say,  $t \geq t_0 > 0$ . Thus  $\varphi(t)$  is monotonically increasing for  $t \geq t_0$ . Then

$$\sum_{j \leq n} b_j \sim \sum_{2 \leq j \leq n} \varphi(j) = \int_2^n \varphi(t) dt + O(\varphi(n)).$$

By the asymptotic relation (2.20), we have

$$\begin{aligned} \int_1^n \varphi(t) dt &= \int_1^n t^\beta (\log t)^\xi \tilde{f}(t)^\alpha dt \\ &\sim (\log \kappa) \int_1^n t^{\beta+1} (\log t)^{\xi-1} \tilde{f}(t)^{\alpha-1} \tilde{f}'(t) dt \\ &\sim \frac{\log \kappa}{\alpha} \int_1^n t^{\beta+1} (\log t)^{\xi-1} d\tilde{f}(t)^\alpha \\ &= \frac{n\varphi(n)}{\alpha \log_\kappa n} + O\left(\int_1^n \frac{\varphi(t)}{t} dt\right), \end{aligned}$$

by an integration by parts. The integral on the right-hand side is easily estimated as follows:

$$\begin{aligned} \int_1^n \frac{\varphi(t)}{t} dt &= O\left(\varphi(qn) \int_1^{qn} t^{-1} dt + \varphi(n) \int_{qn}^n t^{-1} dt\right) \\ &= O(\varphi(n)). \end{aligned}$$

This proves the lemma.  $\square$

PROPOSITION 4.2. *If  $b_n \sim n^\beta (\log n)^\xi \tilde{f}(n)^\alpha$ , where  $\alpha > 1$ ,  $\beta, \xi \in \mathbb{R}$ , then*

$$(4.6) \quad a_n = (1 + O(n^{1-\alpha} (\log n)^{\alpha-1})) \sum_{0 \leq j \leq n} b_j \sim \frac{n}{\alpha \log_\kappa n} b_n.$$

*Proof.* We start with obtaining upper and lower bounds for  $a_n$ . Since  $b_n > 0$  for sufficiently large  $n$ , say,  $n \geq n_0$ . We may, without loss of generality, assume  $b_n \geq 0$  for  $n \geq n_0$  (for, otherwise, we consider  $b'_n := b_n + \max_{j \leq n_0} |b_j|$  and then show the difference between the corresponding  $a'_n$  and  $a_n$  is of order  $\tilde{f}(n)$ ). Then  $a_n \geq 0$  and, by (3.5), we have the lower bound

$$a_n \geq a_{n-1} + b_n \geq \sum_{0 \leq j \leq n} b_j.$$

Now consider the sequence

$$C_n := \frac{a_n}{\sum_{0 \leq j \leq n} b_j} \geq 1 \quad (n \geq 1)$$

and the increasing sequence

$$C_n^* := \max_{1 \leq j \leq n} \{C_j\} \geq 1.$$

Then we have the upper bound

$$a_k \leq C_n^* \sum_{0 \leq j \leq k} b_j$$

for all  $k \leq n$ .

In view of the recurrence relation (3.5), we have

$$a_n \leq C_{n-1}^* \sum_{0 \leq j \leq n} b_j + C_{n-1}^* \sum_{0 \leq j < n} \pi_{n,j} \sum_{0 \leq \ell \leq j} b_\ell.$$

By Lemma 4.1 and Corollary 2.6, we see that there exist an absolute constant  $K > 0$  such that

$$(4.7) \quad \sum_{0 \leq j < n} \pi_{n,j} \sum_{0 \leq \ell \leq j} b_\ell \leq K n^{-\alpha} (\log n)^\alpha \sum_{0 \leq j \leq n} b_j = O(n^{1-\alpha} (\log n)^{\alpha-1} b_n).$$

It follows that

$$a_n \leq C_{n-1}^* (1 + K n^{-\alpha} (\log n)^\alpha) \sum_{0 \leq j \leq n} b_j.$$

By our definition of  $C_n$ , we then have

$$C_n \leq C_{n-1}^* (1 + K n^{-\alpha} (\log n)^\alpha),$$

and

$$C_n^* = \max\{C_{n-1}^*, C_n\} \leq C_{n-1}^* (1 + K n^{-\alpha} (\log n)^\alpha).$$

Consequently,

$$\begin{aligned} C_n^* &\leq C_2^* \prod_{2 \leq j \leq n} (1 + K j^{-\alpha} (\log j)^\alpha) \\ &\leq C_2^* \prod_{j \geq 2} (1 + K j^{-\alpha} (\log j)^\alpha). \end{aligned}$$

Thus we obtain the upper bound

$$a_n \leq C \sum_{0 \leq j \leq n} b_j,$$

where  $C > 0$  is an absolute constant depending only on  $p, \alpha, \beta$ , and  $\xi$ .

With this bound and defining  $\tilde{a}_n := \sum_{0 \leq j < n} \pi_{n,j} a_j$ , we can rewrite the recurrence relation (3.5) as

$$(4.8) \quad \begin{aligned} a_n &= a_{n-1} + \tilde{a}_n + b_n \\ &= \sum_{0 \leq j \leq n} b_j + \sum_{0 \leq k \leq n} \tilde{a}_k. \end{aligned}$$

Now by the estimate (4.7), we see that

$$\begin{aligned} \sum_{0 \leq j \leq n} \tilde{a}_j &= O \left( 1 + \sum_{2 \leq j \leq n} j^{1-\alpha} (\log j)^{\alpha-1} b_j \right) \\ &= O \left( 1 + \varphi(qn) \sum_{2 \leq j \leq qn} j^{1-\alpha} (\log j)^{\alpha-1} + n^{1-\alpha} (\log n)^{\alpha-1} \sum_{qn < j \leq n} b_j \right), \end{aligned}$$

where  $\varphi(t) := t^\beta (\log t)^\xi \tilde{f}(t)^\alpha$ . Observe that

$$\varphi(qn) \sim n^{-\alpha} (\log n)^\alpha b_n \sim n^{-\alpha-1} (\log n)^{\alpha+1} \sum_{0 \leq j \leq n} b_j.$$

Thus

$$\sum_{0 \leq j \leq n} \tilde{a}_j = O \left( n^{1-\alpha} (\log n)^{\alpha-1} \sum_{0 \leq j \leq n} b_j \right).$$

The proof of the Proposition is complete by substituting this estimate into (4.8).  $\square$

Write  $[z^n]A(z)$  for the coefficient of  $z^n$  in the Taylor expansion of  $A(z)$ . Then, in terms of ordinary GFs, the asymptotic transfer (4.6) can be stated alternatively as

$$[z^n]A(z) \sim [z^n] \frac{B(z)}{1-z},$$

(when  $b_n$  satisfies the assumption of Proposition 4.2), which means that the contribution from terms in the sum in (3.6) with  $j \geq 1$  is asymptotically negligible. Roughly, since

$$b_{n,j} := [z^n]B \left( \frac{q^j z}{1 - (1 - q^j)z} \right) = n^{-1} \sum_{1 \leq \ell \leq n} \binom{n}{\ell} q^{j\ell} (1 - q^j)^{n-\ell} \ell b_\ell,$$

we see that  $b_{n,j} = O(q^j b_{\lfloor q^j n \rfloor})$ . We can then give an alternative proof of (4.6) by using (3.6).

By (4.5) and a direct application of Proposition 4.2, we obtain an asymptotic approximation to the variance.

**THEOREM 4.3.** *The variance of  $Y_n$  satisfies*

$$(4.9) \quad \sigma_n^2 \sim C_\sigma n^{-2} (\log_\kappa n)^3 \tilde{f}(n)^2,$$

where  $C_\sigma := p/(2q)$ .

Thus we have

$$\frac{\mathbb{V}(Y_n)}{(\mathbb{E}(Y_n))^2} \sim C_\sigma n^{-2} (\log n)^3.$$

Asymptotics of  $\mathbb{V}(X_n)$  remains open. Monte Carlo simulations (with  $n$  a few hundred) suggested that the ratio  $\mathbb{V}(X_n)/\mathbb{V}(Y_n)$  grows concavely, so that one would expect an order of the form  $n^\beta(\log n)^\xi$  for this ratio for some  $0 < \beta < 1$ . But due to the complexity of the problem, we could not run simulations of larger samples to draw more convincing conclusions.

**5. Asymptotic normality.** We prove in this section that  $Y_n$  is asymptotically normally distributed by the method of moments. Our approach is to start from the recurrence (4.2) for the central moments and the asymptotic estimate (4.9) and then to apply inductively the asymptotic transfer result (Proposition 4.2), similar to that used in our previous papers [35, 37].

THEOREM 5.1. *The distribution of  $Y_n$  is asymptotically normal, namely,*

$$\frac{Y_n - \mu_n}{\sigma_n} \xrightarrow{d} \mathcal{N}(0, 1),$$

where  $\xrightarrow{d}$  denotes convergence in distribution.

We will indeed prove convergence of all moments.

*Proof.* By Frechet and Shohat’s moment convergence theorem (see [49, section 11.4]), it suffices to show that

$$(5.1) \quad M_{n,m} = \mathbb{E}(Y_n - \mu_n)^m \begin{cases} \sim \frac{(m)!}{(m/2)!2^{m/2}} \sigma_n^m & \text{if } m \text{ is even,} \\ = o(\sigma_n^m) & \text{if } m \text{ is odd} \end{cases}$$

for  $m \geq 0$ .

The cases when  $m \leq 2$  having been proved above, we assume  $m \geq 3$ . By the induction hypothesis, we have

$$M_{n,k} = O(\sigma_n^k) = O\left(n^{-k}(\log n)^{3k/2} \tilde{f}^k(n)\right)$$

for  $k < m$ . Then, by (4.4),

$$\begin{aligned} \sum_{0 \leq j < n} \pi_{n,j} M_{j,\ell} \Delta_{n,j}^h &= O\left(M_{\lfloor qn \rfloor, \ell} n^{h/2} \tilde{f}(q^2 n)^h\right) \\ &= O\left(n^{-\ell}(\log n)^{3\ell/2} \tilde{f}(qn)^\ell n^{h/2} \tilde{f}(q^2 n)^h\right) \\ &= O\left(n^{-2\ell-3h/2}(\log n)^{5\ell/2+2h} \tilde{f}(n)^{\ell+h}\right). \end{aligned}$$

It follows (see (4.3)) that, for  $0 \leq \ell < m$ ,

$$\sum_{0 \leq j < n} \pi_{n,j} M_{j,\ell} \Delta_{n,j}^{m-\ell} = O\left(n^{-\ell/2-3m/2}(\log n)^{\ell/2+2m} \tilde{f}(n)^m\right);$$

and, for  $2 \leq k \leq m - 2$  and  $0 \leq \ell \leq m - k$ ,

$$M_{n-1,k} \sum_{0 \leq j < n} \pi_{n,j} M_{j,\ell} \Delta_{n,j}^{m-k-\ell} = O\left(n^{-\ell/2+k/2-3m/2}(\log n)^{\ell/2-k/2+2m} \tilde{f}(n)^m\right).$$

Thus the main contribution to the asymptotics of  $T_{n,m}$  will come from the terms in the second group of sums in (4.3) with  $k = m - 2$  and  $\ell = 0$ . More precisely,

$$T_{n,m} = \binom{m}{2} M_{n-1,m-2} T_{n,2} + O\left(n^{-3/2-m}(\log n)^{3(m+1)/2} \tilde{f}(n)^m\right).$$

Note that  $T_{n,2} \sim 2n(\log_\kappa n)^{-1} \sigma_n^2$ ; see (4.5).



Thus, if  $m$  is even, then, by (4.5) and the induction hypothesis,

$$\begin{aligned} T_{n,m} &\sim \frac{2m!}{((m-2)/2)!2^{m/2}} n^{-1}(\log_{\kappa} n)\sigma_n^m \\ &\sim \frac{2m!}{((m-2)/2)!2^{m/2}} C_{\sigma}^{m/2} n^{-m-1}(\log_{\kappa} n)^{(3m/2+1)} \tilde{f}(n)^m. \end{aligned}$$

Applying the asymptotic transfer result (Proposition 4.2) with  $\alpha = m$ , we obtain

$$\begin{aligned} M_{n,m} &\sim \frac{m!}{(m/2)!2^{m/2}} C_{\sigma}^{m/2} n^{-m}(\log n)^{3m/2} \tilde{f}(n)^m \\ &\sim \frac{m!}{(m/2)!2^{m/2}} \sigma_n^m. \end{aligned}$$

In a similar manner, we can prove that if  $m$  is odd, then

$$M_{n,m} = o(\sigma_n^m).$$

This concludes the proof of (5.1) and the asymptotic normality of  $Y_n$ .  $\square$

**6. The random variables  $Z_n$ .** We briefly consider the random variables defined recursively in (1.10). The major interest is in understanding the robustness of the asymptotic normality when changing the underlying probability distribution from binomial to uniform.

**THEOREM 6.1.** *The mean value of  $Z_n$  satisfies*

$$(6.1) \quad \mathbb{E}(Z_n) = Cn^{-1/4}e^{2\sqrt{n}} \left( 1 + \frac{9}{16\sqrt{n}} + \frac{11}{1536n} + O\left(n^{-3/2}\right) \right),$$

where

$$C := \frac{1}{2}\sqrt{\frac{e}{\pi}} \left( e^{-1} - \int_1^{\infty} \frac{e^{-v}}{v} \right) dv \approx 0.0690646192\dots$$

The limit law of the normalized random variables  $Z_n/\mathbb{E}(Z_n)$  is not normal

$$\frac{Z_n}{\mathbb{E}(Z_n)} \xrightarrow{d} Z,$$

where the distribution of  $Z$  is uniquely characterized by its moment sequence and the GF  $\zeta(y) := \sum_{m \geq 1} \mathbb{E}(Z^m)y^m/(m \cdot m!)$  satisfies the nonlinear differential equation

$$(6.2) \quad y^2\zeta''(y) + y\zeta'(y) - \zeta(y) = y\zeta(y)\zeta'(y)$$

with  $\zeta(0) = \zeta'(0) = 1$ .

*Proof (sketch).* The proof of the theorem is simpler, and we sketch only the major steps.

*Mean value.* First,  $\nu_n := \mathbb{E}(Z_n)$  satisfies the recurrence

$$\nu_n = \nu_{n-1} + \frac{1}{n} \sum_{0 \leq j < n} \nu_j \quad (n \geq 2)$$

with  $\nu_0 = 0$  and  $\nu_1 = 1$ . The GF  $f(z)$  of  $\mathbb{E}(Z_n)$  satisfies the differential equation

$$f'(z) = \frac{2-z}{(1-z)^2} f(z) + \frac{1}{1-z}$$

with the initial condition  $f(0) = 0$ . The first-order differential equation is easily solved, and we obtain the closed-form expression

$$f(z) = -\frac{1}{1-z} + \frac{e^{1/(1-z)}}{1-z} \left( e^{-1} - \int_1^\infty \frac{e^{-v} - e^{-v/(1-z)}}{v} dv \right).$$

From this, the asymptotic approximation (6.1) results from a direct application of the saddle-point method (see Flajolet and Sedgewick’s book [25, Chapter VIII]); see also [23].

*Asymptotic transfer.* For higher moments and the limit law, we are led to consider the following recurrence:

$$(6.3) \quad a_n = a_{n-1} + \frac{1}{n} \sum_{0 \leq j < n} a_j + b_n \quad (n \geq 2)$$

with  $a_0$  and  $a_1$  given. For simplicity, we assume  $a_0 = b_0 = 0$ .

PROPOSITION 6.2. *Assume  $a_n$  satisfies (6.3). If  $b_n \sim cn^\beta \nu_n^\alpha$ , where  $\alpha > 1$  and  $\beta \in \mathbb{R}$ , then*

$$(6.4) \quad a_n \sim \frac{c}{\alpha - \alpha^{-1}} n^{\beta+1/2} \nu_n^\alpha.$$

The proof is similar to that for Proposition 4.2 and is omitted.

*Recurrence and induction.* By Proposition 6.2 and the following recurrence relation for the moment GF  $Q(y) := \mathbb{E}(e^{Z_n y})$

$$Q_n(y) = \frac{Q_{n-1}(y)}{n} \sum_{0 \leq j < n} Q_j(y) \quad (n \geq 2)$$

with  $Q_0(y) = 1$  and  $Q_1(y) = e^y$ , we deduce, by induction using (6.4), that

$$\mathbb{E}(Z_n^m) \sim \zeta_m \nu_n^m \quad (m \geq 1),$$

where

$$(6.5) \quad \zeta_m = \frac{1}{m - m^{-1}} \sum_{1 \leq j < m} \binom{m}{j} \frac{\zeta_j}{j} \zeta_{m-j} \quad (m \geq 2)$$

with  $\zeta_0 = \zeta_1 = 1$ . It follows that the function  $\zeta(y) := \sum_{m \geq 1} \zeta_m y^m / (m \cdot m!)$  satisfies the differential equation (6.2).

*Unique determination of the distribution.* First, by a simple induction we can show, by (6.5), that  $\zeta_m \leq cm!K^m$  for a sufficiently large  $K > 0$ . This is enough for justifying the unique determination. Instead of giving the details, it is more interesting to note that the nonlinear differential equation (6.2) represents another typical case for which the asymptotic behavior of its coefficients ( $\mathbb{E}(Z^m)$  for large  $m$ ) necessitates the use of the psi-series method recently developed in [10]. We can show, by the approach used there, that

$$\mathbb{E}(Z^m) = m \cdot m! \rho^{-m} \left( 2 + \frac{2}{3m^2} + O(m^{-3}) \right),$$

where  $\rho > 0$  is an effectively computable constant. Note that there is no term of the form  $m^{-1}$  in the expansion, a typical situation when the psi-series method applies; see [10].  $\square$

**Concluding remarks.** The approach we used in this paper is of some generality and is amenable to other quantities. We conclude this paper with a few examples and a list of some concrete applications where the scale  $n^{c \log n}$  also appears.

First, the expected number of independent sets in a random graph (under the  $\mathcal{G}_{n,p}$  model), as given in (1.8), satisfies the recurrence ( $\bar{J}_n := J_n + 1$ )

$$\bar{J}_n = \bar{J}_{n-1} + \sum_{0 \leq k < n} \binom{n-1}{k} q^k p^{n-1-k} \bar{J}_k \quad (n \geq 1)$$

with  $\bar{J}_0 = 1$ . Thus the Poisson GF  $\tilde{f}(z) := e^{-z} \sum_{n \geq 0} \bar{J}_n z^n / n!$  satisfies the equation

$$\tilde{f}'(z) = \tilde{f}(qz)$$

with  $\tilde{f}(0) = 1$ . The modified Laplace transform then satisfies the functional equation

$$\tilde{f}^*(s) = 1 + s \tilde{f}^*(qs),$$

which, by iteration, leads to the closed-form expression

$$\tilde{f}^*(s) = \sum_{j \geq 0} q^{j(j-1)/2} s^j.$$

Thus all analysis as in section 2 applies with  $F$  and  $G$ , there replaced by

$$F(s) := \sum_{j \in \mathbb{Z}} q^{j(j-1)/2} s^j, \quad G(u) := q^{\{u\}^2 + \{u\}/2} F(q^{-\{u\}}).$$

We obtain, for example,

$$J_n = \frac{G\left(\log_\kappa \frac{n}{\log_\kappa n}\right)}{\sqrt{2\pi}} \cdot \frac{n^{1/\log_\kappa n + 1/2}}{\log_\kappa n} \exp\left(\frac{\left(\log \frac{n}{\log_\kappa n}\right)^2}{2 \log \kappa}\right) \left(1 + O\left(\frac{(\log \log n)^2}{\log n}\right)\right).$$

Note that, alternatively, one may also use the explicit sum expression (1.8) and apply the saddle-point method for characterizing the asymptotics of  $J_n$ .

The same approach also applies to the pantograph equation

$$\Phi'(z) = a\Phi(qz) + \Psi(z) \quad (a > 0),$$

with  $\Phi(0)$  and  $\Psi(z)$  given, for  $\Psi(z)$  satisfying properties that can be easily imposed.

Other extensions will be discussed elsewhere. We conclude with some other algorithmic, combinatorial, and analytic contexts where  $n^{c \log n}$  appears:

- Algorithmics: isomorphism testing (see [4, 30, 34, 52, 61]), autocorrelations of strings (see [31, 60]), information theory (see [2]), random digital search trees (see [20]), population recovery (see [67]), and asymptotics of recurrences (see [45, 54]);
- Combinatorics: partitions into powers (see [15, 50]; see also [27] for a brief historical account and more references), palindromic compositions (see [41]), combinatorial number theory (see [7, 47]), and universal tree of minimum complexity (see [11, 29]);
- Probability: log-normal distribution (see [42]), renewal theory (see [65, 66]), and total positivity (see [43]);

- Algebra: commutative ring theory (see [8]) and semigroups (see [46, 58, 62]);
- Analysis: pantograph equations (see [38, 44]), eigenfunctions of operators (see [63]), geometric partial differential equations (see [18]), and  $q$ -difference equations (see [3, 9, 19, 57, 70, 71]).

This list shows to some extent the generality of the seemingly uncommon scale  $n^{c \log n}$ ; also it suggests the possibly nontrivial connections between instances in various areas, whose clarification in turn may lead to further development of more useful tools such as those in this paper.

**Acknowledgments.** We thank the referees for helpful comments and suggestions.

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