

Limit distribution of the coefficients of polynomials with only unit roots

Hsien-Kuei Hwang*
Institute of Statistical Science,
Institute of Information Science
Academia Sinica
Taipei 115
Taiwan

Vytas Zacharovas†
Dept. Mathematics & Informatics
Vilnius University
Naugarduko 24, Vilnius
Lithuania

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Abstract

We consider sequences of random variables whose probability generating functions have only roots on the unit circle, which has only been sporadically studied in the literature. We show that the random variables are asymptotically normally distributed if and only if the fourth central and normalized (by the standard deviation) moment tends to 3, in contrast to the common scenario for polynomials with only real roots for which a central limit theorem holds if and only if the variance is unbounded. We also derive a representation theorem for all possible limit laws and apply our results to many concrete examples in the literature, ranging from combinatorial structures to numerical analysis, and from probability to analysis of algorithms.

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1 Introduction

The close connection between the location of the zeros of a function (or a polynomial) and the distribution of its coefficients has long been the subject of extensive study; typical examples include the order of an entire function and its zeros in Analysis, and the limit distribution of the coefficients of polynomials when all roots are real in Combinatorics, Probability and Statistical Physics. We address in this paper the situation when the roots of the sequence of probability generating functions all lie on the unit circle. While one may convert the situation

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with only unimodular zeros to that with only real zeros by a suitable change of variables, such *root-unitary polynomials* turn out to have many fascinating properties due mainly to the boundedness of all zeros. In particular, we show that the fourth normalized central moments are (asymptotically) always bounded between 1 and 3, the limit distribution being Bernoulli if they tend to 1 and Gaussian if they tend to 3.

Although this class of polynomials does not have a standard name, we will refer to them as, following [25] and for convention, root-unitary polynomials. Other related terms include *self-inversive* (zeros are symmetric with respect to unit circle), *reciprocal* or *self-reciprocal* ($P(z) = z^n P(z^{-1})$) or *palindromic* ($a_j = a_{n-j}$), *uni-modular* (all coefficients of modulus one), etc., when $P(z) = \sum_{0 \leq j \leq n} a_j z^j$ is a polynomial of degree n .

Unit roots of polynomials play a very special and important role in many scientific and engineering disciplines, notably in statistics and signal processing where the unit root test decides if a time series variable is non-stationary. On the other hand, many nonparametric statistics are closely connected to partitions of integers, which lead to generating functions whose roots all lie on the unit circle. We will discuss many examples in Sections 4 and 5. Another famous example is the Lee-Yang partition function for the Ising model, which has stimulated a widespread interest in the statistical-physical literature since the 1950's.

While there is a large literature on polynomials with only real zeros, the distribution of the coefficients of root-unitary polynomials has only been sporadically studied; more references will be given below. It is well known that for polynomials with nonnegative coefficients whose roots are all real, one can decompose the polynomials into products of linear factors, implying that the associated random variables are expressible as sums of independent Bernoulli random variables. Thus one obtains a *Gaussian limit law for the coefficients if and only if the variance tends to infinity*; see Pitman's survey paper [38] for more information and for finer estimates; see also [4, 18]. A representative example is the Stirling numbers of the second kind for which Harper [20] showed that the generating polynomials have only real roots¹; he also established the asymptotic normality of these numbers by proving that the variance tends to infinity. For more examples and information on polynomials with only real roots, see [6], [38] and the references therein. See also [18], [21], [41] for different extensions.

Our first main result states that if we restrict the range where the roots of the polynomials $P_n(z)$ can occur to the unit circle $|z| = 1$, then the asymptotic normality of X_n defined by the coefficients is determined by the limiting behavior of its fourth normalized central moment. *Throughout this paper, write $X_n^* := (X_n - \mathbb{E}(X_n)) / \sqrt{\mathbb{V}(X_n)}$.*

Theorem 1.1. *Let $\{X_n\}$ be a sequence of discrete random variables whose probability generating functions $\mathbb{E}(z^{X_n})$ are polynomials of degree n with all roots ρ_j lying on the unit circle $|\rho_j| = 1$. Then the following properties hold.*

– (Bounds for the fourth normalized central moment)

$$1 \leq \mathbb{E}(X_n^*)^4 < 3 \quad (n \geq 1). \quad (1)$$

– (Asymptotic normality) *The sequence of random variables $\{X_n^*\}$ converges in distribution (and with all moments) to the standard normal law $\mathcal{N}(0, 1)$ if and only if*

$$\mathbb{E}(X_n^*)^4 \rightarrow 3 \quad (n \rightarrow \infty). \quad (2)$$

¹The fact that the Stirling polynomials (of the second kind) have only real roots had been known long before Harper [20]; see for example [14]; in addition, Bell [3] wrote (without providing reference) that all results of d'Ocagne's paper [14] were already obtained thirty years before him by a number of English authors.

– (Asymptotic Bernoulli distribution) The sequence $\{X_n^*\}$ converges to Bernoulli random variable assuming the two values -1 and 1 with equal probabilities if and only if

$$\mathbb{E}(X_n^*)^4 \rightarrow 1 \quad (n \rightarrow \infty). \quad (3)$$

This theorem implies that Gaussian and Bernoulli distributions are in a certain sense extremal limit laws for the distribution of X_n , maximizing and minimizing asymptotically the value of the fourth moment $\mathbb{E}(X_n^*)^4$, respectively, with other limit laws lying in between.

If we define the *kurtosis* of a distribution X by²

$$\text{Kurt}(X) := \frac{\mathbb{E}(X - \mathbb{E}(X))^4}{\mathbb{V}(X)^2}, \quad (4)$$

then the theorem states that the kurtosis of X_n always lies between 1 and 3, and whether the limit law is normal or Bernoulli depends on the limiting kurtosis being 3 or 1. As can be seen by applying the Cauchy-Schwarz inequality, the kurtosis of a random variable is always greater than or equal to 1, but can be arbitrarily large in general. Note that the same kurtosis condition (1) for asymptotic normality also appeared previously in a completely different context in [35].

While most previous papers dealing with root-unitary polynomials require to check all moments when the limit law is normal, our proof here relies instead on a closer examination of the root structure, leading particularly to the optimal conditions.

A standard example where a Gaussian limit law arises is the number of inversions in random permutations (or Kendall's τ -statistic)

$$P_{\binom{n}{2}}(z) = \prod_{1 \leq k \leq n} \frac{1 + z + \dots + z^{k-1}}{k}.$$

A straightforward calculation shows that the kurtosis has the form

$$3 - \frac{9(6n^2 + 15n + 16)}{25n(n-1)(n+1)},$$

which implies the asymptotic normality by Theorem 1.1; see [16], Section 4 for more details and examples.

On the other hand, a Bernoulli limit law results from the simple example

$$P_{2n}(z) = \frac{1 + z^{2n}}{2}.$$

It is then natural to ask to which limit laws other than normal and Bernoulli can the sequence of random variables X_n^* converge. The simplest such example is the uniform distribution

$$P_{2n}(z) = \frac{1 + z + z^2 + \dots + z^{2n}}{2n + 1},$$

or, more generally, the finite sum of uniform distributions.

Observe that the moment generating functions of the above three distributions have the following representations.

²This definition of kurtosis differs from the usual one by a factor of -3 .

- Normal distribution: $e^{s^2/2}$;
- Bernoulli distribution (assuming ± 1 with equal probability):

$$\frac{e^s + e^{-s}}{2} = \cos(is) = \prod_{k \geq 1} \left(1 + \frac{4s^2}{(2k-1)^2 \pi^2} \right);$$

- Uniform distribution (with zero mean and unit variance):

$$\frac{1}{2\sqrt{3}} \int_{-\sqrt{3}}^{\sqrt{3}} e^{xs} dx = \frac{\sin(\sqrt{3}is)}{\sqrt{3}is} = \prod_{k \geq 1} \left(1 + \frac{3s^2}{k^2 \pi^2} \right).$$

Here we used the well-known expansions (see [48])

$$\cos s = \prod_{k \geq 1} \left(1 - \frac{4s^2}{(2k-1)^2 \pi^2} \right), \quad \text{and} \quad \frac{\sin s}{s} = \prod_{k \geq 1} \left(1 - \frac{s^2}{k^2 \pi^2} \right).$$

We show that these are indeed special cases of a more general representation theorem for the limit laws.

Theorem 1.2. *Let $\{X_n\}$ be a sequence of random variables whose probability generating functions are polynomials with only roots of modulus one. If the sequence $\{X_n^*\}$ converges to some limit distribution X , then the moment generating function of X is finite and has the infinite-product representation*

$$\mathbb{E}(e^{Xs}) = e^{qs^2/2} \prod_{k \geq 1} \left(1 + \frac{q_k}{2} s^2 \right), \quad (5)$$

where q and the sequence $\{q_k\}$ are all non-negative numbers such that

$$q + \sum_{k \geq 1} q_k = 1.$$

The above examples show that $q_k = \frac{8}{\pi^2(2k-1)^2}$ for Bernoulli distribution and $q_k = \frac{6}{\pi^2 k^2}$ for the uniform distribution. More examples will be discussed below.

It remains open to characterize infinite-product representations of the form (5) that are themselves the moment generating functions of limit laws of root-unitary polynomials. On the other hand, many sufficient criteria for root-unitarity have been proposed in the literature; see, for example, the books [33, 45] and the recent papers [29, 46] for more information and references.

This paper is organized as follows. We first prove Theorem 1.1 in the next section when n is even, and then modify the proof to cover polynomials of odd degrees. Theorem 1.2 is then proved in Section 3. We then apply the results to many concrete examples from the literature: Section 4 for normal limit laws and Section 5 for non-normal laws. A very simple and effective means (first used by Euler) of computing the zeros of the limiting moment generating function is sketched in Appendix.

2 Moments and the two extremal limit distributions

For convenience, we begin by considering (general) polynomials of even degree with all their roots lying on the unit circle

$$P_{2n}(z) = \sum_{0 \leq k \leq 2n} p_k z^k,$$

where $p_k \geq 0$. To avoid triviality, we assume that not all p_k 's are zero. Observe that if $|\rho| = 1$ and $P(\rho) = 0$, then $P(\bar{\rho}) = 0$. If $\rho = 1$, then its multiplicity must be even since all other roots can be grouped in pairs and are symmetric with respect to the real line. Thus our polynomials can be factored as

$$P_{2n}(z) = \prod_{1 \leq j \leq n} (z - \rho_j)(z - \bar{\rho}_j),$$

where $|\rho_j| = 1$ for $j = 1, \dots, n$. This factorization implies that root-unitary polynomials enjoy several properties.

Lemma 2.1. *A root-unitary polynomial of even degree $2n$ is self-inversive and self-reciprocal (or palindromic), namely,*

$$p_{n-k} = p_{n+k} \quad (0 \leq k \leq n).$$

Proof. By replacing z by $1/z$, we get

$$\begin{aligned} \sum_{0 \leq k \leq 2n} p_{2n-k} z^k &= z^{2n} P_{2n}(1/z) = \prod_{1 \leq j \leq n} (1 - z\rho_j)(1 - z\bar{\rho}_j) \\ &= \prod_{1 \leq j \leq n} (z - \rho_j)(z - \bar{\rho}_j) = P_{2n}(z) = \sum_{0 \leq k \leq 2n} p_k z^k. \end{aligned}$$

Taking the coefficients of z^k on both sides, we obtain $p_{2n-k} = p_k$ for $0 \leq k \leq 2n$, which proves the lemma. ■

2.1 Random variables, moments and cumulants

Since the coefficients of $P_{2n}(z)$ are nonnegative, we can define a random variable X_{2n} by

$$\mathbb{E}(z^{X_{2n}}) = \frac{P_{2n}(z)}{P_{2n}(1)}.$$

For convenience, we write $\rho_j = e^{i\phi_j}$ since $|\rho_j| = 1$. Then

$$(z - \rho_j)(z - \bar{\rho}_j) = 1 - 2z \cos \phi_j + z^2.$$

It follows that

$$\mathbb{E}(z^{X_{2n}}) = \prod_{1 \leq j \leq n} \frac{1 - 2z \cos \phi_j + z^2}{2(1 - \cos \phi_j)}. \quad (6)$$

Note that $\phi_j \neq 0$ for $1 \leq j \leq n$ since $P_{2n}(1) > 0$.

It turns out that the mean values of such random variables are identically n .

Lemma 2.2. *For $n \geq 1$*

$$\mathbb{E}(X_{2n}) = n. \quad (7)$$

Proof. By (6), take derivative with respect to z and then substitute $z = 1$. ■

The relation (7) indeed holds more generally for self-inversive polynomials; see, for example, [45].

Corollary 2.3. *All odd central moments of X_{2n} are zero*

$$\mathbb{E}(X_{2n} - n)^{2m+1} = 0 \quad (m = 0, 1, \dots).$$

Proof. This follows from the symmetry of the coefficients p_k . ■

For even moments, we look at the cumulants, which are defined as

$$\mathbb{E}(e^{(X_{2n}-n)s}) = \exp\left(\sum_{m \geq 1} \frac{\kappa_m(n)}{m!} s^m\right),$$

where $\kappa_{2m+1}(n) = 0$.

Lemma 2.4. *The $2m$ -th cumulant $\kappa_{2m}(n)$ of X_{2n} is given by*

$$\kappa_{2m}(n) = (2m)! \sum_{1 \leq k \leq m} \frac{(-1)^{k-1}}{k 2^k} h_{m,k} S_{n,k} \quad (m \geq 1), \quad (8)$$

where $2^{2k} \sinh^{2k}(s/2) = \sum_{m \geq k} h_{m,k} s^{2m}$, with $h_{k,k} = 1$, and

$$S_{n,k} := \sum_{1 \leq j \leq n} \frac{1}{(1 - \cos \phi_j)^k}.$$

Proof. By (6), we have

$$\log \frac{1 - 2e^s \cos \phi + e^{2s}}{2(1 - \cos \phi)} = s + \log \left(1 + 2 \frac{\sinh^2(s/2)}{1 - \cos \phi}\right).$$

Thus

$$\log \mathbb{E}(e^{(X_{2n}-n)s}) = \sum_{1 \leq j \leq n} \log \left(1 + 2 \frac{\sinh^2(s/2)}{(1 - \cos \phi_j)}\right),$$

which implies (8). ■

2.2 Variance and kurtosis

In particular, we obtain, from (8),

$$\sigma_n^2 := \mathbb{V}(X_{2n}) = \kappa_2(n) = \sum_{1 \leq j \leq n} \frac{1}{1 - \cos \phi_j}. \quad (9)$$

Lemma 2.5. *The variance satisfies the inequalities*

$$\frac{n}{2} \leq \sigma_n^2 \leq n^2. \quad (10)$$

Proof. The lower bound follows from (9) and the inequality $1 - \cos \phi_j \leq 2$. The upper bound is also straightforward

$$\mathbb{V}(X_{2n}) = \frac{1}{P_{2n}(1)} \sum_{0 \leq k \leq 2n} p_k (k - n)^2 \leq n^2,$$

which shows that the distance between any root of $P_{2n}(z)$ to the point 1 is always larger than c/n , where $c > 0$ is an absolute constant. ■

On the other hand, by the elementary inequalities

$$\frac{2}{\pi^2} t^2 \leq 1 - \cos t \leq \frac{t^2}{2} \quad (t \in [-\pi, \pi]), \quad (11)$$

we have

$$2 \leq \frac{\sigma_n^2}{\sum_{1 \leq j \leq n} \phi_j^{-2}} \leq \frac{\pi^2}{2}.$$

We now turn to the fourth central moment. Define

$$\omega_n := \frac{S_{n,2}}{S_{n,1}^2} = \frac{1}{\sigma_n^4} \sum_{1 \leq j \leq n} \frac{1}{(1 - \cos \phi_j)^2}. \quad (12)$$

Lemma 2.6. (i) For $n \geq 1$,

$$1 \leq \text{Kurt}(X_{2n}) \leq 3 - \frac{1}{2\sigma_n^2} < 3. \quad (13)$$

(ii)

$$\text{Kurt}(X_{2n}) \rightarrow 3 \quad \text{iff} \quad \omega_n \rightarrow 0. \quad (14)$$

Proof. By definition and by (8),

$$\text{Kurt}(X_{2n}) = 3 + \frac{\kappa_4(n)}{\sigma_n^4} = 3 + \sigma_n^{-2} - 3\omega_n.$$

Now

$$\sigma_n^{-2} - 3\omega_n = -\frac{1}{\sigma_n^4} \sum_{1 \leq j \leq n} \frac{2 + \cos \phi_j}{(1 - \cos \phi_j)^2} \leq -\frac{1}{2\sigma_n^4} \sum_{1 \leq j \leq n} \frac{1}{1 - \cos \phi_j} = -\frac{1}{2} \sigma_n^{-2} < 0,$$

proving the upper bound of (13). On the other hand, since $1/\sigma_n \leq \sqrt{2/n}$ (by (10)), we see that (14) also holds. It remains to prove the lower bound of (13), which results directly from the Cauchy-Schwarz inequality

$$1 = \mathbb{E} \left(\frac{X_{2n} - n}{\sigma_n} \right)^2 \leq \left(\mathbb{E} \left(\frac{X_{2n} - n}{\sigma_n} \right)^4 \right)^{1/2}.$$

■

By (11), we can replace the condition $\omega_n \rightarrow 0$ by

$$\sum_{1 \leq j \leq n} \phi_j^{-4} = o\left(\left(\sum_{1 \leq j \leq n} \phi_j^{-2}\right)^2\right).$$

This means that whether the limit law is normal depends on how slow the unit roots approach unity. Roughly, if ϕ_j does not tend to zero too fast, say $\phi_j \gg \sqrt{j}/n$, then the limit law is always normal; see Figures 1–7 for an illustration. On the other hand, the condition $\text{Kurt}(X_n) \rightarrow 3$ is equivalent to $\kappa_4(n)/\kappa_2^2(n) \rightarrow 0$; the latter condition is in many cases easier to manipulate; see Section 4.

Note that (13) proves (1) when n is even.

2.3 Estimates for the moment generating functions

Lemma 2.7. *For all $s \in \mathbb{C}$ such that $|s| \leq \min\{\sigma_n, \omega_n^{-1/4}\}/4$, we have*

$$\mathbb{E}(e^{(X_{2n}-n)s/\sigma_n}) = \exp\left(\frac{s^2}{2} + O\left(\frac{|s|^3}{\sigma_n} + \omega_n |s|^4\right)\right). \quad (15)$$

Proof. By (6),

$$\log \mathbb{E}(e^{X_{2n}s/\sigma_n}) = \sum_{1 \leq j \leq n} \log\left(1 + (e^{s/\sigma_n} - 1) + \frac{(e^{s/\sigma_n} - 1)^2}{2(1 - \cos \phi_j)}\right)$$

Note that, by (9),

$$\sigma_n^2 \geq \max_{1 \leq j \leq n} \frac{1}{1 - \cos \phi_j}.$$

From the definition (12) of ω_n , we also have

$$\frac{1}{\sigma_n^4} \max_{1 \leq j \leq n} \frac{1}{(1 - \cos \phi_j)^2} \leq \omega_n,$$

which means that

$$\max_{1 \leq j \leq n} \frac{1}{1 - \cos \phi_j} \leq \sigma_n^2 \sqrt{\omega_n}. \quad (16)$$

Thus

$$\left|e^{s/\sigma_n} - 1 + \frac{(e^{s/\sigma_n} - 1)^2}{2(1 - \cos \phi_j)}\right| \leq e^{2|s|/\sigma_n} \left(\frac{|s|}{\sigma_n} + |s|^2 \sqrt{\omega_n}\right)$$

Note that for s satisfying our assumption $|s| \leq \min\{\sigma_n, \omega_n^{-1/4}\}/4$ the right-hand side does not exceed $e^{1/2}(1/4 + 1/4^2) < 1$. Thus we can use the Taylor expansion of $\log(1 + w)$ and obtain

$$\begin{aligned} & \log\left(1 + (e^{s/\sigma_n} - 1) + \frac{(e^{s/\sigma_n} - 1)^2}{2(1 - \cos \phi_j)}\right) \\ &= \frac{s}{\sigma_n} + \frac{s^2}{2\sigma_n^2(1 - \cos \phi_j)} + O\left(\frac{|s|^3}{\sigma_n^3(1 - \cos \phi_j)} + \frac{|s|^4}{\sigma_n^4(1 - \cos \phi_j)^2} + \frac{|s|^6}{\sigma_n^6(1 - \cos \phi_j)^3}\right). \end{aligned}$$

By (16)

$$\frac{|s|^6}{\sigma_n^6(1 - \cos \phi_j)^3} \leq \frac{|s|^6 \sqrt{\omega_n}}{\sigma_n^4(1 - \cos \phi_j)^2}.$$

It follows, after summing over all j , that

$$\log \mathbb{E}(e^{X_{2n}s/\sigma_n}) = \frac{s^2}{2} + \frac{ns}{\sigma_n} + O\left(\frac{|s|^3}{\sigma_n} + (|s|^4 + |s|^6 \sqrt{\omega_n})\omega_n\right).$$

Now if $|s| \leq \min\{\sigma_n, \omega_n^{-1/4}\}/4$, then $\omega_n^{3/2}|s|^6 \leq \omega_n|s|^4/16$, and this proves (15). ■

Lemma 2.8. For $s \in \mathbb{R}$, the inequality

$$\mathbb{E}(e^{(X_{2n}-n)s/\sigma_n}) \leq \exp\left(\frac{3}{2}s^2 e^{2s/\sigma_n}\right) \quad (17)$$

holds.

Proof. By (6) and the elementary inequality $1 + y \leq e^y$ for real y , we obtain

$$\begin{aligned} \mathbb{E}(z^{X_{2n}}) &\leq \prod_{1 \leq j \leq n} \exp\left(z - 1 + \frac{(z-1)^2}{2(1-\cos\phi_j)}\right) \\ &= e^{n(z-1) + \sigma_n^2(z-1)^2/2}. \end{aligned}$$

Thus

$$\mathbb{E}(e^{(X_{2n}-n)s/\sigma_n}) \leq \exp\left(\frac{n}{2\sigma_n^2} s^2 e^{s/\sigma_n} + \frac{s^2}{2} e^{2s/\sigma_n}\right),$$

and (17) follows from the inequality $n/\sigma_n^2 \leq 2$. ■

2.4 Normal limit law

We now prove the second part of Theorem 1.1 in the case of polynomials of even degree, namely, $\{(X_n - n)/\sigma_n\}$ converges in distribution and with all moments to the standard normal distribution if and only if

$$\text{Kurt}(X_{2n}) = \mathbb{E}\left(\frac{X_{2n} - n}{\sigma_n}\right)^4 \rightarrow 3.$$

Proof. Consider first the sufficiency part. By (14), $\omega_n \rightarrow 0$, and we can apply the estimate (15), implying the convergence in distribution of $(X_{2n} - n)/\sigma_n$ to $\mathcal{N}(0, 1)$.

On the other hand, by Lemma 2.8,

$$\mathbb{E}(e^{(X_{2n}-n)s/\sigma_n}) = \sum_{m \geq 0} \left(\frac{X_{2n} - n}{\sigma_n}\right)^{2m} \frac{s^{2m}}{(2m)!} \leq e^{\frac{3}{2}s^2 e^{2s/\sigma_n}}.$$

Taking $s = 1$, we conclude that all normalized central moments of X_{2n} are bounded above by

$$\mathbb{E}\left(\frac{X_{2n} - n}{\sigma_n}\right)^{2m} \leq (2m)! e^{\frac{3}{2}e^{2/\sigma_n}}.$$

Thus we also have convergence of all moments.

For the necessity, we see that if $\{(X_{2n} - n)/\sigma_n\}$ converges in distribution to $\mathcal{N}(0, 1)$, then the fact that the moments of $(X_{2n} - n)/\sigma_n$ are all bounded implies that all the normalized central moments of X_{2n} converge to the moments of the standard normal distribution; in particular, the kurtosis converges to 3. ■

2.5 Bernoulli limit law

We now examine the case when the fourth moment converges to the smallest possible value, that is

$$\text{Kurt}(X_{2n}) \rightarrow 1. \quad (18)$$

Note that

$$\mathbb{V}\left(\frac{X_{2n}-n}{\sigma_n}\right)^2 = \mathbb{E}\left(\left(\frac{X_{2n}-n}{\sigma_n}\right)^2 - 1\right)^2 = \mathbb{E}\left(\frac{X_{2n}-n}{\sigma_n}\right)^4 - 1.$$

If (18) holds, then by Chebyshev's inequality, we see that

$$\mathbb{P}\left(\frac{X_{2n}-n}{\sigma_n} \in (-1-\varepsilon, -1+\varepsilon) \cup (1-\varepsilon, 1+\varepsilon)\right) \rightarrow 1,$$

for any $\varepsilon > 0$. By symmetry of the random variable $X_{2n} - n$

$$\mathbb{P}\left(\frac{X_{2n}-n}{\sigma_n} \in (-1-\varepsilon, -1+\varepsilon)\right) = \mathbb{P}\left(\frac{X_{2n}-n}{\sigma_n} \in (1-\varepsilon, 1+\varepsilon)\right).$$

We conclude that the distributions of $(X_{2n} - n)/\sigma_n$ converge to a Bernoulli distribution that assumes the two values 1 and -1 with equal probability.

2.6 Polynomials of odd degree

To complete the proof of Theorem 1.1, we now address the situation of odd-degree polynomials.

Assume $Q_{2n-1}(z)$ is a root-unitary polynomial of degree $2n - 1$ with non-negative coefficients. If we multiply it by the factor $1 + z$, then the resulting polynomial

$$P_{2n}(z) = (1 + z)Q_{2n-1}(z)$$

remains root-unitary with non-negative coefficients. This means that the moment generating functions of the corresponding random variables $\mathbb{E}(e^{Y_{2n-1}s}) := Q_{2n-1}(e^s)/Q_{2n-1}(1)$ and $\mathbb{E}(e^{X_{2n}s}) := P_{2n}(e^s)/P_{2n}(1)$ are connected by the identity

$$\mathbb{E}(e^{X_{2n}s}) = \frac{1 + e^s}{2} \mathbb{E}(e^{Y_{2n-1}s}).$$

This leads to the relation

$$X_{2n} \stackrel{d}{=} Y_{2n-1} + B, \quad (19)$$

where B is independent of Y_{2n-1} and takes the values 0 and 1 with equal probability. Thus

$$\begin{aligned} \mathbb{E}(Y_{2n-1}) &= \mathbb{E}(X_{2n}) - \frac{1}{2} = n - \frac{1}{2}, \\ \mathbb{V}(Y_{2n-1}) &= \mathbb{V}(X_{2n}) - \frac{1}{4} = \sigma_n^2 - \frac{1}{4}, \end{aligned} \quad (20)$$

and

$$\mathbb{E}(Y_{2n-1} - \mathbb{E}(Y_{2n-1}))^4 = \mathbb{E}(X_{2n} - n)^4 - \frac{3}{2}\sigma_n^2 + \frac{5}{16}. \quad (21)$$

Thus we obtain

$$\text{Kurt}(Y_{2n-1}) \leq \frac{\sigma_n^4}{(\sigma_n^2 - \frac{1}{4})^2} \text{Kurt}(X_{2n}) - \frac{\frac{3}{2}\sigma_n^2 - \frac{5}{16}}{(\sigma_n^2 - \frac{1}{4})^2},$$

which, by (13), is bounded above by

$$\frac{\sigma_n^4}{(\sigma_n^2 - \frac{1}{4})^2} \left(3 - \frac{1}{\sigma_n^2} \right) - \frac{\frac{3}{2}\sigma_n^2 - \frac{5}{16}}{(\sigma_n^2 - \frac{1}{4})^2} = 3 - \frac{1}{\sigma_n^2} - \frac{6\sigma_n^2 - 1}{\sigma_n^2(4\sigma_n^2 - 1)^2} \leq 3 - \sigma_n^{-2} < 3.$$

Thus the kurtosis is bounded above by 3; the lower bound follows from the same Cauchy-Schwarz inequality used in the even-degree cases.

On the other hand, since (again by (19))

$$\frac{X_{2n} - \mathbb{E}(X_{2n})}{\sqrt{\mathbb{V}(X_{2n})}} \stackrel{d}{=} \frac{\sqrt{\mathbb{V}(Y_{2n-1})}}{\sqrt{\mathbb{V}(X_{2n})}} \cdot \frac{Y_{2n-1} - \mathbb{E}(Y_{2n-1})}{\sqrt{\mathbb{V}(Y_{2n-1})}} + \frac{B - \frac{1}{2}}{\sqrt{\mathbb{V}(X_{2n})}},$$

we have, by (20),

$$\frac{X_{2n} - \mathbb{E}(X_{2n})}{\sqrt{\mathbb{V}(X_{2n})}} \stackrel{d}{=} \frac{Y_{2n-1} - \mathbb{E}(Y_{2n-1})}{\sqrt{\mathbb{V}(Y_{2n-1})}} \left(1 + O\left(\frac{1}{\sqrt{n}}\right) \right) + O(\sigma_n^{-1}). \quad (22)$$

The last identity implies that both sides converge to the same limit law.

Assume that the kurtosis of Y_{2n-1} satisfies

$$\text{Kurt}(Y_{2n-1}) \rightarrow 3.$$

Then, by (21), we obtain

$$\text{Kurt}(X_{2n}) = \left(\frac{\mathbb{V}(Y_{2n-1})}{\mathbb{V}(X_{2n})} \right)^2 \text{Kurt}(Y_{2n-1}) + O(\sigma_n^{-1}).$$

Thus the left-hand side also tends to 3 and, consequently, X_{2n} is asymptotically normally distributed. The asymptotic distribution of X_{2n} then implies, by (22), that of Y_{2n-1} .

The proof for the Bernoulli case is similar and is omitted.

3 The infinite-product representation for general limit laws

We first prove Theorem 1.2 in this section, and then mention some of its consequences.

3.1 Proof of Theorem 1.2

The proof of Theorem 1.2 relies on the Hadamard factorization theorem (see [48, Ch. 8]; see also [34] for a similar context). Indeed, assume that $(X_{2n} - n)/\sigma_n$ converges in distribution to some limit law X , then the inequality (17) implies that

$$|\mathbb{E}(e^{Xs})| \leq e^{3|s|^2/2} \quad (s \in \mathbb{C}).$$

In other words, it is an entire function of order 2. Hadamard's factorization theorem then implies that such a function can be represented as an infinite product

$$\mathbb{E}(e^{Xs}) = e^{As^2 + Bs} \prod_{\rho} \left(1 - \frac{s}{\rho} \right) e^{s/\rho},$$

where ρ ranges over all zeros of the function of the left-hand side. On the other hand, the fact that all zeroes of the functions $\mathbb{E}(e^{(X_{2n-n})s/\sigma_n})$ are symmetrically located on the imaginary line implies the same property for $\mathbb{E}(e^{Xs})$. This yields

$$\mathbb{E}(e^{Xs}) = e^{As^2 + Bs} \prod_{k \geq 1} \left(1 + \frac{s^2}{t_k^2}\right),$$

for some real sequence $t_k > 0$. Now $\mathbb{E}(X) = 0$ implies that $B = 0$. Also $\mathbb{E}(X^2) = 1$ leads to

$$A + \sum_{k \geq 1} t_k^{-2} = 1.$$

Denoting by $q = 2A$ and $q_k = 2/t_k^2$, we obtain the representation (5).

3.2 An alternative proof of Theorem 1.2

We give in this subsection an elementary proof that does not rely on complex analysis. It suffices to consider only the sequence of polynomials of even degree. The symmetry of distribution of the limit law X follows from the symmetry of coefficients of polynomials $P_{2n}(z)$. The inequality (17) for the moment generating function of $(X_{2n} - n)/\sigma_n$ implies that the moment generating function of the limit distribution X is also finite, and thus X is uniquely determined by its moments. This means that the sequence $\{(X_{2n} - n)/\sigma_n\}$ converges in distribution to X as $n \rightarrow \infty$ if and only if

$$\mathbb{E}\left(\frac{X_{2n} - n}{\sigma_n}\right)^m \rightarrow \mathbb{E}(X^m) \quad (m \geq 0),$$

as $n \rightarrow \infty$. Thus the cumulant $\bar{\kappa}_m(n)$ of $(X_{2n} - n)/\sigma_n$ of order m also converges to the cumulant of X of order m for $m \geq 1$. Note that $\bar{\kappa}_{2m+1}(n) = 0$ for $m \geq 0$ and (see (8))

$$\bar{\kappa}_{2m}(n) = \sigma_n^{-2m} \kappa_{2m}(n) = \frac{(2m)!}{\sigma_n^{2m}} \sum_{1 \leq k \leq m} \frac{(-1)^{k-1}}{k 2^k} h_{m,k} S_{n,k}.$$

Since $S_{n,k} \leq \sigma_n^{2k}$, we deduce that

$$\frac{\bar{\kappa}_{2m}(n)}{(2m)!} = \frac{(-1)^{m-1}}{m 2^m} \cdot \frac{S_{n,m}}{\sigma_n^{2m}} + O(\sigma_n^{-2}),$$

for any fixed m . Now $\sigma_n \rightarrow \infty$, we conclude that

$$\frac{\kappa_{2m}}{(2m)!} = \lim_{n \rightarrow \infty} \frac{\bar{\kappa}_{2m}(n)}{(2m)!} = \frac{(-1)^{m-1}}{m 2^m} \lim_{n \rightarrow \infty} \frac{S_{n,m}}{\sigma_n^{2m}}. \quad (23)$$

We now introduce the distribution function

$$F_n(x) := \sum_{\frac{1}{\sigma_n^2(1-\cos\phi_j)} \leq x} \frac{1}{\sigma_n^2(1-\cos\phi_j)},$$

with support in the unit interval. Then

$$\frac{S_{n,N}}{\sigma_n^{2N}} = \int_0^1 x^{N-1} dF_n(x).$$

The fact that the left-hand side of the above expression has a limit (23) implies that the corresponding sequence of distribution functions $F_n(x)$ also converges weakly to some limit distribution function $F(x)$. Therefore

$$\lim_{n \rightarrow \infty} \frac{S_{n,N}}{\sigma_n^{2N}} = \int_0^1 x^{N-1} dF(x),$$

which implies that the cumulants of the limit distribution X can be expressed as

$$\frac{\bar{\kappa}_{2m}}{(2m)!} = \lim_{n \rightarrow \infty} \frac{\bar{\kappa}_{2m}(n)}{(2m)!} = \frac{(-1)^{m-1}}{m2^m} \lim_{n \rightarrow \infty} \frac{S_{n,m}}{\sigma_n^{2m}} = \frac{(-1)^{m-1}}{m2^m} \int_0^1 x^{m-1} dF(x).$$

It follows that

$$\begin{aligned} \mathbb{E}(e^{Xs}) &= \exp \left(\sum_{m \geq 1} \frac{(-1)^{m-1} s^{2m}}{m2^m} \int_0^1 x^{m-1} dF(x) \right) \\ &= \exp \left(\int_0^1 \frac{\log(1 + xs^2/2)}{x} dF(x) \right). \end{aligned} \quad (24)$$

Note that the distribution function $F_n(x)$ has no more than $\lfloor 1/\varepsilon \rfloor$ points of discontinuity in the interval $[\varepsilon, 1]$ if $\varepsilon > 0$. Thus the weak limit $F(x)$ of the sequence of $F_n(x)$ has the same property: $F(x)$ has no more than $\lfloor 1/\varepsilon \rfloor$ points of discontinuity q_k in the interval $[\varepsilon, 1]$, where q_k is the limit of certain points of discontinuity of function $F_n(x)$. This means that $F(x)$ is a distribution function of the form

$$F(x) = \begin{cases} q + \sum_{q_k \leq x} q_k, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases}$$

where $q_k > 0$ with $\sum_{k \geq 1} q_k = 1 - q$. Here q equals the jump of the function $F(x)$ at zero. Thus

$$\int_0^1 \frac{\log(1 + xs^2/2)}{x} dF(x) = \frac{q}{2} s^2 + \sum_{k \geq 1} \log \left(1 + \frac{q_k}{2} s^2 \right).$$

Substituting this expression into (24), we obtain (5). This completes the proof of Theorem 1.2.

3.3 Implications of the infinite-product factorization

By (5),

$$\kappa_{2m} = (2m)! \frac{(-1)^{m-1}}{m2^m} \sum_{j \geq 1} q_j^m \quad (m \geq 1).$$

This yields the sign-alternating property for the sequence $\{\kappa_{2m}\}$.

Corollary 3.1. *If X is not the normal law, then all even cumulants are non-zero and have alternating signs*

$$(-1)^{m-1} \kappa_{2m} > 0 \quad (m \geq 1).$$

Corollary 3.2.

$$1 \leq \mathbb{E}(X^4) \leq 3.$$

Proof. By (5),

$$\mathbb{E}(X^4) = 3 \left(1 - \sum_{j \geq 1} q_j^2 \right), \quad (25)$$

which implies the upper bound; the lower bound follows directly from Cauchy-Schwarz inequality $1 = \mathbb{E}(X^2) \leq \sqrt{\mathbb{E}(X^4)}$. ■

Corollary 3.3. *The standard normal distribution is the only distribution for which the fourth moment reaches the maximum value 3 in the class of distributions that are the limits of random variables whose probability generating functions are root-unitary polynomials ; similarly, the Bernoulli distribution assuming ± 1 with probability $1/2$ each is the only distribution whose fourth moment reaches the minimum value 1 in the same class of distributions.*

Proof. Note that the standard normal law corresponds to the choices $q = 1$ and $q_j \equiv 0$, the first part of the corollary follows then from (25).

For the lower bound, assume that Y is a symmetric distribution such that $\mathbb{E}(Y) = 0$ and $\mathbb{E}(Y^2) = \mathbb{E}(Y^4) = 1$. Then

$$\mathbb{V}(Y^2) = \mathbb{E}(Y^2 - 1)^2 = \mathbb{E}(Y^4 - 2Y^2 + 1) = 0.$$

This means that Y can only assume two values $\mathbb{P}(Y \in \{-1, 1\}) = 1$. The symmetry of Y now implies that Y assumes the values 1 and -1 with equal probabilities. ■

Remark 3.4. *The uniqueness of the standard normal and Bernoulli laws also implies that a sequence of random variables $\{X_n\}$ converges to normal or Bernoulli if and only if their kurtoses converge to 3 or to 1, respectively. This provides an alternative proof of the last two statements of Theorem 1.1.*

4 Applications. I. Normal limit law

We consider in this section applications of our results in the situations when the limit law is normal.

4.1 A simple framework

Our starting point is the polynomials of the form

$$P_n(z) = \frac{(1 - z^{b_1})(1 - z^{b_2}) \dots (1 - z^{b_N})}{(1 - z^{a_1})(1 - z^{a_2}) \dots (1 - z^{a_N})}, \quad (26)$$

where a_j, b_j are non-negative integers that may depend themselves on N and

$$n := \sum_{1 \leq j \leq N} (b_j - a_j).$$

We assume that $P_n(z)$ has only nonnegative coefficients. Such a simple form arises in a large number of diverse contexts some of which will be examined below. In particular, it was studied in the recent paper [9].

We now consider a sequence of random variables X_n defined by

$$\mathbb{E}(z^{X_n}) = \frac{P_n(z)}{P_n(1)}.$$

We have

$$\frac{P_n(e^s)}{P_n(1)} = \exp\left(\sum_{m \geq 1} \frac{\kappa_{N,m}}{m!} s^m\right),$$

where

$$\kappa_{N,m} = \frac{(-1)^m}{m} B_m \sum_{1 \leq j \leq N} (b_j^m - a_j^m) \quad (m \geq 1),$$

the B_m 's being the Bernoulli numbers. Note that $B_{2m+1} = 0$ for $m \geq 1$.

An application of Theorem 1.1 yields the following result.

Theorem 4.1. *The sequence of the random variables $(X_n - \mathbb{E}(X_n))/\sqrt{\mathbb{V}(X_n)}$ converges to the standard normal distribution if and only if the following cumulant condition holds*

$$\lim_{N \rightarrow \infty} \frac{\kappa_{N,4}}{\kappa_{N,2}^2} = \frac{144}{120} \lim_{N \rightarrow \infty} \frac{\sum_{1 \leq j \leq N} (b_j^4 - a_j^4)}{(\sum_{1 \leq j \leq N} (b_j^2 - a_j^2))^2} = 0. \quad (27)$$

The cumulant condition largely simplifies the sufficient condition given in [9], where they require the convergence of all cumulants (following the proof used in [42])

$$\frac{\kappa_{N,2m}}{\kappa_{N,2}^m} \rightarrow 0 \quad (m \geq 2).$$

See also [23] for a related framework.

4.2 Applications of Theorem 4.1

Theorem 4.1 can be applied to a large number of examples. Many other examples related to Poincaré polynomials, rank statistics, and integer partitions can be found in the literature; see, for example, [1, 2, 13, 50] and the references therein.

Inversions in permutations The generating polynomial for the number of inversions in a permutation of n elements (or Kendall's τ statistic) is given by

$$\prod_{1 \leq j \leq N} \frac{1 - z^j}{1 - z}.$$

In this case, the cumulant condition (27) has the form

$$\frac{\sum_{1 \leq j \leq N} (j^4 - 1)}{(\sum_{1 \leq j \leq N} (j^2 - 1))^2} = O(N^{-1}).$$

Thus the number of inversions in random permutations is asymptotically normally distributed; see [16], [42]; see also [12, 30, 32].

Number of inversions in Stirling permutations In this case, we have the polynomial (see [37])

$$\prod_{1 \leq j \leq N} \frac{1 - z^{r+(j-1)r^2}}{1 - z^r} \quad (r \geq 1),$$

and the cumulant condition (27) is of order

$$\frac{\kappa_{N,4}}{\kappa_{N,2}^2} = \frac{\sum_{0 \leq j < N} ((r + jr^2)^4 - 1)}{(\sum_{0 \leq j < N} ((r + jr^2)^2 - 1))^2} = O(N^{-1}).$$

Consequently, the number of inversions in random Stirling permutations is asymptotically normally distributed.

Gaussian polynomials The generating function for the number $p(n, m, j)$ of partitions of integer j into at most m parts, each $\leq n$, is given by (see e.g. [2])

$$\sum_{0 \leq j \leq Nm} p(N, m, j) z^j = \prod_{1 \leq j \leq N} \frac{1 - z^{j+m}}{1 - z^j}.$$

Then the cumulant condition has the form

$$\frac{\sum_{1 \leq j \leq N} ((m+j)^4 - j^4)}{(\sum_{1 \leq j \leq N} ((m+j)^2 - j^2))^2} = O\left(\frac{1}{m} + \frac{1}{N}\right).$$

This means that the coefficients of Gaussian polynomials are normally distributed if both $N, m \rightarrow \infty$; see [31, 47]. More examples can be found in [2].

Mahonian statistics In this case the polynomials are equal to the general q -multinomial coefficients (see [7] and [8])

$$P_n(z) = \frac{\prod_{1 \leq j \leq a_1 + \dots + a_m} (1 - z^j)}{\prod_{1 \leq j \leq m} \prod_{1 \leq i \leq a_j} (1 - z^i)},$$

where $n = \sum_{2 \leq k \leq m} a_k \sum_{1 \leq j < k} a_j$. By symmetry, we can assume that $a_1 \geq \dots \geq a_m$. Then the cumulant condition (27) becomes

$$\frac{\sum_{1 \leq j \leq a_1 + \dots + a_m} i^4 - \sum_{1 \leq j \leq m} \sum_{1 \leq i \leq a_j} i^4}{\left(\sum_{1 \leq j \leq a_1 + \dots + a_m} i^2 - \sum_{1 \leq j \leq m} \sum_{1 \leq i \leq a_j} i^2\right)^2} = \frac{f_4(a_1 + \dots + a_m) - \sum_{1 \leq j \leq m} f_4(a_j)}{(f_2(a_1 + \dots + a_m) - \sum_{1 \leq j \leq m} f_2(a_j))^2},$$

where $f_2(x) = (2x^3 + 3x^2 + x)/6$ and $f_4(x) = (6x^5 + 15x^4 + 10x^3 - x)/30$. By induction, $(a_1 + \dots + a_m)^k - a_1^k - \dots - a_m^k$ is nonnegative and is nondecreasing in $k \geq 1$. Thus the right-hand side is bounded above by

$$\begin{aligned} & \frac{9 \cdot 31}{30} \frac{(a_1 + \dots + a_m)^5 - a_1^5 - \dots - a_m^5}{((a_1 + \dots + a_m)^3 - a_1^3 - \dots - a_m^3)^2} \\ &= O\left(\frac{a_1 + \dots + a_m}{\sum_{1 \leq i < j \leq m} a_i a_j}\right) = O\left(\frac{a_1 + \dots + a_m}{a_1(a_2 + a_3 + \dots + a_m)}\right) \\ &= O\left(\frac{1}{a_2 + a_3 + \dots + a_m} + \frac{1}{a_1}\right), \end{aligned}$$

where we use the estimates

$$(a_1 + \cdots + a_m)^3 - a_1^3 - \cdots - a_m^3 \asymp (a_1 + \cdots + a_m) \sum_{1 \leq i < j \leq m} a_i a_j,$$

$$(a_1 + \cdots + a_m)^5 - a_1^5 - \cdots - a_m^5 \asymp (a_1 + \cdots + a_m)^3 \sum_{1 \leq i < j \leq m} a_i a_j.$$

Thus we arrive at the same conditions as those given in [7].

$$a_1 \rightarrow \infty \quad \text{and} \quad a_2 + a_3 + \cdots + a_m \rightarrow \infty,$$

for the asymptotic normality of the coefficients of $P_n(z)$ when $a_1 \geq a_2 \geq \cdots \geq a_m$. See also [13, pp. 128–129] for a closely related structure and results.

Generalized q -Catalan numbers The generating function has the form (see [19])

$$\prod_{2 \leq j \leq N} \frac{1 - z^{(m-1)N+j}}{1 - z^j},$$

and the cumulant condition (27) also holds

$$\frac{\sum_{2 \leq j \leq N} (((m-1)N+j)^4 - j^4)}{(\sum_{2 \leq j \leq N} (((m-1)N+j)^2 - j^2))^2} \leq \frac{\sum_{2 \leq j \leq N} (2mN)^4}{(\sum_{2 \leq j \leq N} (m-1)^2 N^2)^2} = O(N^{-1}),$$

which means that the generalized q -Catalan numbers are asymptotically normally distributed, uniformly for all $m \geq 2$. This was previously proved in [9]. For more information, see [19].

Sums of uniform discrete distributions Let X_n be the sum of N independent, integer-valued random variables

$$X_n := J_1 + J_2 + \cdots + J_N,$$

where J_k is a uniform distribution on the set $\{0, 1, 2, \dots, d_k - 1\}$ with $d_k \geq 2$, and $n = \sum_{1 \leq j \leq N} (d_j - 1)$. Then the corresponding probability generating function $\mathbb{E}(z^{X_n})$ is equal, up to a normalizing constant, to

$$P_n(z) = \prod_{1 \leq j \leq N} \frac{1 - z^{d_j}}{1 - z},$$

which means that X_n is asymptotically normal if and only if

$$\frac{\sum_{1 \leq j \leq N} (d_j^4 - 1)}{(\sum_{1 \leq j \leq N} (d_j^2 - 1))^2} \rightarrow 0.$$

Since by our assumption $d_j \geq 2$, we have $d_j - 1 \asymp d_j$ and thus we can simplify our necessary and sufficient condition for asymptotic normality as

$$\frac{d_1^4 + d_2^4 + \cdots + d_N^4}{(d_1^2 + d_2^2 + \cdots + d_N^2)^2} \rightarrow 0 \quad (N \rightarrow \infty). \quad (28)$$

Note that d_j here can depend on N . The continuous version of this problem with J_k being uniformly distributed on the intervals $[0, d_j]$ was considered in [36]. The corresponding necessary and sufficient condition obtained in this paper was

$$\frac{\max_{1 \leq j \leq N} d_j}{\sqrt{d_1^2 + d_2^2 + \cdots + d_N^2}} \rightarrow 0$$

which is equivalent to condition (28).

Number of inversions in bimodal permutations A permutation $\sigma = (s_1, s_2, \dots, s_n)$ of n numbers $1, 2, 3, \dots, n$ is said to be of a shape $(i, k - j, j, l)$ if the first i numbers in the permutation are decreasing $s_1 > s_2 > \dots > s_i$, the next $k - j$ numbers are increasing $s_{i+1} > s_{i+2} > \dots > s_{i+k-j}$, then followed by j increasing and l decreasing numbers. Assume that σ is chosen with equal probability among all permutations of shape $(i, k - j, j, l)$. Then its number of inversions becomes a random variable $I_n = I_n(i, k - j, j, l)$. The probability generating function of I_n is, up to some constant, of the form (see [5])

$$P_n(i, k, l, j; z) = z^{\binom{i}{2} + \binom{j}{2}} \left(\prod_{1 \leq v \leq i} \frac{1 - z^{k+v}}{1 - z^v} \right) \left(\prod_{1 \leq v \leq l} \frac{1 - z^{k+i+v}}{1 - z^v} \right) \left(\prod_{1 \leq v \leq j} \frac{1 - z^{k-j+v}}{1 - z^v} \right).$$

The random variables I_n are asymptotically normally distributed if

$$\frac{\sum_{v=1}^i ((k+v)^4 - v^4) + \sum_{v=1}^l ((k+i+v)^4 - v^4) + \sum_{v=1}^j ((k-j+v)^4 - v^4)}{\left(\sum_{v=1}^i ((k+v)^2 - v^2) + \sum_{v=1}^l ((k+i+v)^2 - v^2) + \sum_{v=1}^j ((k-j+v)^2 - v^2) \right)^2} \rightarrow 0,$$

which is equivalent to

$$\frac{ik(k+i)^3 + l(k+i)(k+i+l)^3 + j(k^4 - j^4)}{(ik(k+i) + l(k+i)(k+i+l) + j(k^2 - j^2))^2} \rightarrow 0.$$

If we assume that the parameters i, j, k, l are proportional to some parameter t , that is $i = \lfloor \alpha t \rfloor, j = \lfloor \beta t \rfloor, k = \lfloor \gamma t \rfloor, l = \lfloor \delta t \rfloor$, where $\alpha, \beta, \gamma, \delta > 0$ and $\alpha + \gamma + \delta = 1$, then the above condition is satisfied and as a consequence I_n is asymptotically normally distributed as $t \rightarrow \infty$. This fact has been proved in [5] by the method of moments.

Rank statistics Many test statistics based on ranks lead to explicit generating functions that are of the form (26), and thus the corresponding limit distribution can be dealt with by the tools we established. In particular, we have the following correspondence between test statistics and combinatorial structures; see [22, 50] for more information.

Kendall's τ	Inversions in permutations
Mann-Whitney test	Gaussian polynomials
Jonckheere-Terpstra test	Mahonian statistics

On the other hand, the Wilcoxon signed rank test (see [52]) leads to the probability generating function of the form

$$\prod_{1 \leq j \leq N} \frac{1 + z^j}{2},$$

which admits a straightforward generalization to (see [50] for details)

$$\prod_{1 \leq j \leq N} \frac{1 + z^{a_j}}{2},$$

where the a_j 's can be any real numbers. When they are all nonnegative integers, we see, by a similar argument leading to the condition (27), that the associated random variables are asymptotically normally distributed if and only if

$$\frac{a_1^4 + \dots + a_N^4}{(a_1^2 + \dots + a_N^2)^2} \rightarrow 0,$$

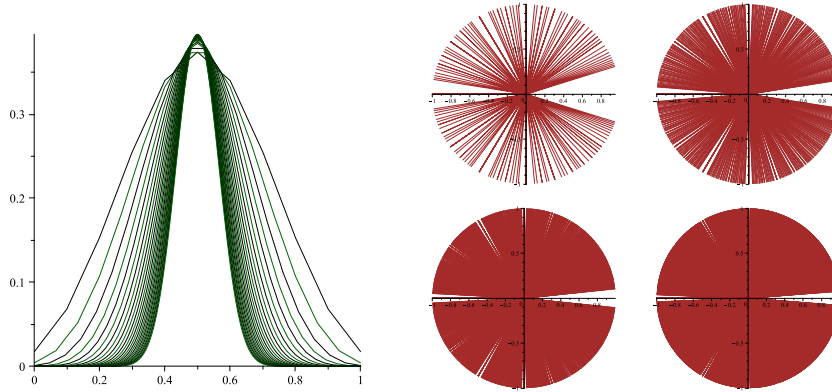


Figure 1: *Normalized histograms (left) of Kendall's τ statistic (multiplied by standard variation) for $n = 5, \dots, 30$ and the distributions of the zeros (right) for $n = 20, 40, 60, 80$.*

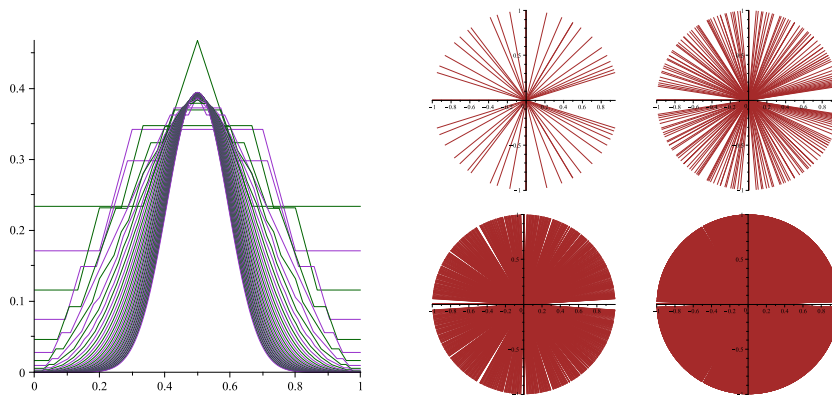


Figure 2: *Normalized histograms (left) of Wilcoxon's test statistic (multiplied by standard variation) for $n = 3, \dots, 40$ and the distributions of the zeros (right) for $n = 10, 20, 50, 100$.*

as $N \rightarrow \infty$. In particular, this applies to Wilcoxon's test ($a_j = j$) and to Policello and Hettmansperger's test ($a_j = \min\{2j, N + 1\}$; [39]). See Figures 1–3 for an illustration of the distribution of the coefficients and the distribution of the zeros.

4.3 Turán-Fejér polynomials

The class of polynomials we consider here (see (29) below) is of interest for several reasons. First, they lead to asymptotically normally distributed random variables but do not have the finite-product form (26). Second, they provide natural examples with non-normal limit laws when the second parameter varies. Finally, they have a concrete interpretation in terms of the partitioning cost of some variants of quicksort, one of the most widely used sorting algorithms; see [10, 44].

Fejér [15] studied the Cesàro summation of the geometric series defined by

$$F_{n,k}(z) := \sum_{0 \leq j \leq n} F_{j,k-1}(z) \quad (k \geq 1),$$

with

$$F_{n,0}(z) := \sum_{0 \leq j \leq n} z^j,$$

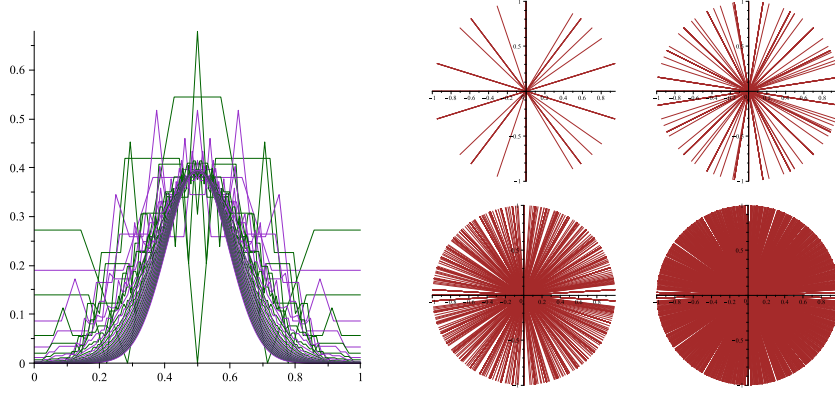


Figure 3: Normalized histograms (left) of Policello and Hettmansperger's test statistic (multiplied by standard variation) for $n = 3, \dots, 40$ and the distributions of the zeros (right) for $n = 10, 20, 50, 100$.

and Turán [49] proved that all $F_{n,k}^{(k)}(z)$ are root-unitary for $0 \leq k \leq n$. We characterize all possible limit laws for the random variables defined via the coefficients of $F_{n,k}^{(k)}(z)$ for $0 \leq k \leq n$.

By the relation

$$F_{n,k}(z) = [w^n] \frac{1}{(1-w)^{k+1}(1-zw)},$$

where $[w^n]f(w)$ denotes the coefficient of w^n in the Taylor expansion of $f(w)$, we have

$$\begin{aligned} F_{n,k}^{(k)}(z) &= [w^n] \frac{1}{(1-w)^{k+1}} \cdot \frac{k!w^k}{(1-zw)^{k+1}} \\ &= k! \sum_{0 \leq j \leq n-k} \binom{j+k}{k} \binom{n-j}{k} z^j. \end{aligned}$$

Normalizing this polynomial, we obtain

$$P_{n,k}(z) := \sum_{0 \leq j \leq n-k} \frac{\binom{j+k}{k} \binom{n-j}{k}}{\binom{n+k+1}{2k+1}} z^j, \quad (29)$$

which gives rise to a sequence of probability generating functions of random variables, say $Z_{n,k}$. Note that

$$z^k P_{n-k-1,k}(z) = \sum_{k \leq j \leq n-k-1} \frac{\binom{j}{k} \binom{n-1-j}{k}}{\binom{n}{2k+1}} z^j,$$

which arises in the analysis of quicksort using the median of $2k + 1$ elements; see [10, 44].

Lemma 4.2. For $m \geq 0$

$$\mathbb{E}(Z_{n,k}^m) = \sum_{0 \leq \ell \leq m} S(m, \ell) \ell! \frac{\binom{k+\ell}{k} \binom{n+k+1}{2k+\ell+1}}{\binom{n+k+1}{2k+1}}, \quad (30)$$

where $S(m, \ell)$ denotes the Stirling numbers of the second kind. In particular,

$$\mathbb{E}(Z_{n,k}) = \frac{n-k}{2} \quad \text{and} \quad \mathbb{V}(Z_{n,k}) = \frac{(n-k)(n+k+2)}{4(2k+3)}. \quad (31)$$

Proof. By (29), the relation

$$j^m = \sum_{0 \leq \ell \leq m} S(m, \ell) j \cdots (j - \ell + 1),$$

and the combinatorial identity

$$\sum_{0 \leq j \leq n-k} \binom{j+k}{k} \binom{n-j}{k} \binom{j}{\ell} = \binom{k+\ell}{k} \binom{n+k+1}{2k+\ell+1},$$

(easily proved by convolution), we deduce (30). ■

Theorem 4.3. *The random variables $Z_{n,k}$ are asymptotically normally distributed if and only if both k and $n-k$ tend to infinity. If $0 \leq k = O(1)$, then the limit law is a Beta distribution*

$$\frac{Z_{n,k}}{n} \xrightarrow{d} \text{Beta}(k, k).$$

If $1 \leq \ell := n-k = O(1)$, then the limit law is a binomial distribution

$$Z_{n,k} \xrightarrow{d} \text{Binom}(\ell; \frac{1}{2}).$$

Proof. By (31), the variance tends to infinity if and only if $n-k \rightarrow \infty$ ($0 \leq k \leq n$). Also we obtain, by (30),

$$\frac{\mathbb{E}(Z_{n,k} - \frac{n-k}{2})^4}{\mathbb{V}(Z_{n,k})^2} - 3 = -\frac{2(3n^2 + 6n + k^2 + 4k + 6)}{(n-k)(n+k+2)(2k+5)} = O\left(\frac{n}{k(n-k)}\right).$$

The asymptotic normality then follows. We can indeed obtain a local limit theorem by straightforward calculations from (29).

When $k = O(1)$, we have, by (29),

$$\frac{\mathbb{E}(Z_{n,k}^m)}{n^m} \rightarrow \frac{(k+m)!(2k+1)!}{k!(2k+m+1)!} \quad (m \geq 0),$$

implying that the moment generating function of the limit law satisfies

$$\mathbb{E}(e^{Z_k s}) = \frac{(2k+1)!}{k!} \sum_{m \geq 0} \frac{(k+m)!}{m!(2k+m+1)!} s^m = \frac{(2k+1)!}{k!k!} \int_0^1 x^k (1-x)^k e^{xs} dx,$$

a Beta distribution. Note that we can express the moment generating function in terms of Bessel functions as

$$\mathbb{E}(e^{(Z_k - 1/2)s}) = \left(\frac{is}{4}\right)^{-k-1/2} \Gamma(k + \frac{3}{2}) J_{k+\frac{1}{2}}(is/2) = \prod_{j \geq 1} \left(1 + \frac{s^2}{4\zeta_{k+1/2,j}^2}\right),$$

where J_α denotes the Bessel function and the $\zeta_{\alpha,j}$'s denote the positive zeros of $J_\alpha(z)$ arranged in increasing order. By considering $2(Z_k - \frac{1}{2})\sqrt{2k+3}$, we obtain (5) with $q_j = 2(2k+3)/\zeta_{k+1/2,j}^2$ and kurtosis (see (4))

$$\text{Kurt}(Z_k) = 3 - \frac{6}{2k+5},$$

which lies between $\frac{9}{5}$ and 3.

On the other hand, when $\ell := n - k = O(1)$, we have, by (30),

$$P_{n,k}(z) \rightarrow \left(\frac{1+z}{2}\right)^\ell,$$

a binomial distribution. Note that we have the factorization

$$\mathbb{E}(e^{(X-\ell/2)s/\sqrt{\ell/4}}) = \prod_{j \geq 1} \left(1 + \frac{4s^2}{(2j-1)^2\pi^2\ell}\right)^\ell;$$

also this is a degenerate case because the degree $n - k$ of $P_{n,k}$ does not tend to infinity. ■

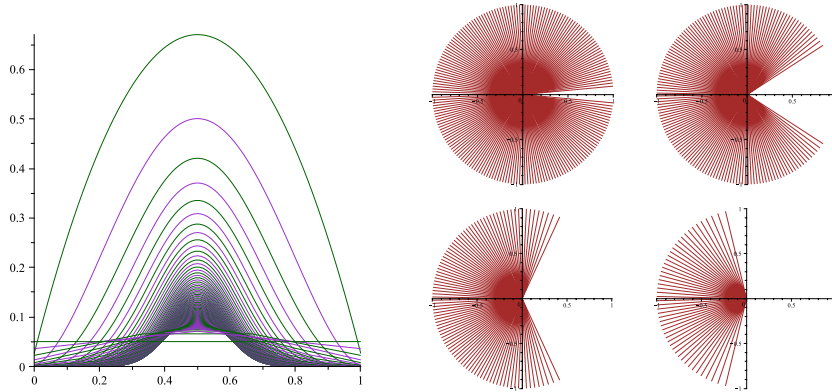


Figure 4: Normalized histograms (left) of $Z_{n,k}$ (scaled by standard variation) for $n = 100$ and $k = 1, \dots, 99$ (from inside out), and the distributions of the zeros (right) for $n = 200$ and $k = 5, 50, 100, 150$.

5 Applications II. Non-normal limit laws

In addition to the extremal cases of the Turán-Fejér polynomials, we consider in this section more root-unitary polynomials whose coefficients lead to a limit distribution that is not Gaussian.

5.1 Reimer's polynomials

In the course of investigating the remainder theory of finite difference, Reimer [40] proved, as a side result, that the polynomials

$$R_{n,m}(y) := \sum_{0 \leq j \leq n} \binom{n}{j} y^j \int_j^{j+1} |t(t-1)\cdots(t-n-1)|^m dt.$$

have only unit roots. We consider the distribution of the coefficients of $R_{n,m}(y)$.

For simplicity, we consider only $m = 1$ and write $R_n = R_{n,1}$. Define the random variables X_n by

$$\mathbb{E}(y^{X_n}) := \frac{R_n(y)}{R_n(1)}.$$

Let

$$A_k := [z^k] \frac{z}{\log(1-z)} \quad (k \geq 0).$$

These numbers are (up to sign) known under the name of Cauchy numbers; see [11, pp. 293–294]. See also the recent paper [26] for a detailed study of these numbers.

Lemma 5.1. *For $n \geq 1$*

$$\mathbb{E}(y^{X_n}) = 12 \sum_{0 \leq j \leq n} \binom{n}{j} y^{n-j} (1-y)^j A_{j+2}. \quad (32)$$

Proof. We have

$$\begin{aligned} R_n(y) &= \sum_{0 \leq j \leq n} \binom{n}{j} (-1)^{n+1+j} y^j \int_j^{j+1} t(t-1) \cdots (t-n-1) dt \\ &= (n+2)! (-1)^{n+1} \sum_{0 \leq j \leq n} \binom{n}{j} (-1)^j y^j \int_j^{j+1} \binom{t}{n+2} dt \\ &= (n+2)! (-1)^{n+1} [z^{n+2}] \sum_{0 \leq j \leq n} \binom{n}{j} (-1)^j y^j \int_j^{j+1} (1+z)^t dt \\ &= (n+2)! (-1)^{n+1} [z^{n+1}] \sum_{0 \leq j \leq n} \binom{n}{j} (-1)^j y^j \frac{(1+z)^j}{\log(1+z)} \\ &= (n+2)! [z^{n+1}] \frac{(1-(1-z)y)^n}{\log(1-z)}. \end{aligned} \quad (33)$$

In particular

$$R_n(1) = (n+2)! [z] \frac{1}{\log(1-z)} = \frac{(n+2)!}{12},$$

and (32) follows. ■

The sequence $\{A_k\}$ can be recursively computed by $A_0 = -1$ and

$$A_k = - \sum_{0 \leq j < k} \frac{A_j}{k+1-j} \quad (k \geq 1).$$

All A_k 's are positive except A_0 .

Lemma 5.2. *The moments of X_n satisfy*

$$\mathbb{E}(X_n^m) = \sum_{0 \leq k \leq m} \tilde{A}_k S(m, k) n(n-1) \cdots (n-k+1) \quad (m \geq 0), \quad (34)$$

where

$$\tilde{A}_k := 12 \sum_{0 \leq \ell \leq k} \binom{k}{\ell} (-1)^\ell A_{\ell+2}. \quad (35)$$

In particular,

$$\mathbb{E}(X_n) = \frac{n}{2}, \quad \mathbb{V}(X_n) = \frac{n}{60} (4n+11).$$

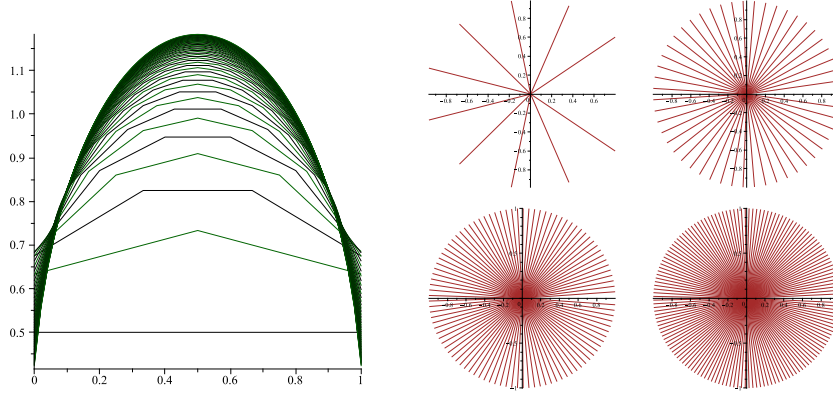


Figure 5: Normalized histograms (left) of Reimer's polynomials (scaled by standard variation) for $n = 1, \dots, 50$ and the distributions of the zeros (right) for $n = 10, 50, 100, 150$.

Proof. By taking m -th derivative with respect to y and then substituting $y = 1$ in (33), we obtain

$$\mathbb{E}(X_n(X_n - 1) \cdots (X_n - m + 1)) = \tilde{A}_m n(n-1) \cdots (n-m+1),$$

which yields (34) since by definition

$$\mathbb{E}(X_n^m) = \sum_{0 \leq k \leq m} S(m, k) \mathbb{E}(X_n(X_n - 1) \cdots (X_n - k + 1)).$$

■

Theorem 5.3. *The sequence of random variables $\{X_n/n\}$ converges in distribution to X whose m -th moment equals \tilde{A}_m (defined in (35)).*

Proof. By (34), $\mathbb{E}(X_n^m) \sim \tilde{A}_m n^m$. Since $A_k = O(1/k)$, we see that $\tilde{A}_m = O(2^m)$, implying that such a moment sequence determines uniquely a distribution. ■

The limit law has the moment generating function

$$\begin{aligned} \mathbb{E}(e^{Xs}) &= 12 \sum_{m \geq 0} \frac{s^m}{m!} \sum_{0 \leq j \leq m} \binom{m}{j} (-1)^j A_{j+2} \\ &= 12e^s \sum_{j \geq 0} \frac{A_{j+2}}{j!} (-s)^j. \end{aligned}$$

Note that $\text{Kurt}(X) = \frac{55}{28}$. It is less obvious how to characterize all zeros of $\mathbb{E}(e^{Xs})$. We sketch in Appendix a simple idea due to Euler to compute the locations of the first few zeros.

When $m \geq 2$, the same arguments apply but the technicalities become more involved.

5.2 Chung-Feller's arcsine law

The classical Chung-Feller theorem states that the number of positive terms W_n of the sums $S_n = X_1 + \cdots + X_n$, where X_i takes ± 1 with probability $1/2$ each, has the probability

$$\mathbb{P}(W_n = k) = \binom{2k}{k} \binom{2n-2k}{n-k} 4^{-n} \quad (k = 0, \dots, n).$$

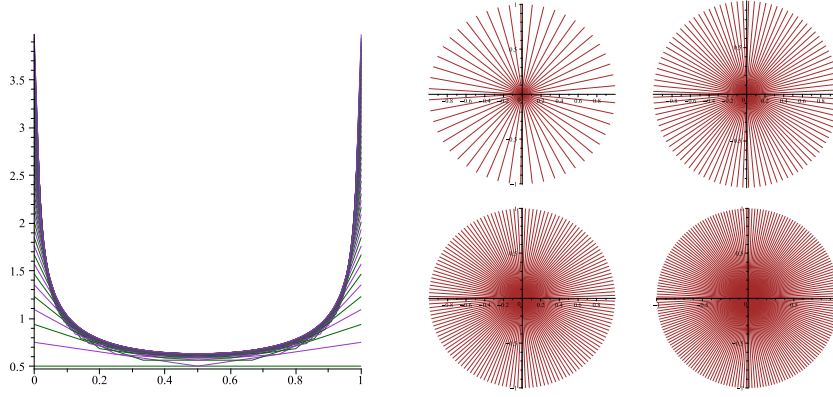


Figure 6: Normalized histograms (left) of W_n (scaled by standard variation) for $n = 1, \dots, 50$ and the distributions of the zeros of $\mathbb{E}(z^{W_n})$ (right) for $n = 50, 100, 150, 200$.

The limit distribution is an arcsine law (see [17, §III.4])

$$\frac{W_n}{n} \xrightarrow{d} W, \quad \text{where} \quad \mathbb{P}(W < x) = \frac{2}{\pi} \arcsin \sqrt{x}.$$

The corresponding probability generating function is a polynomial with only unit roots. Indeed, following the same proof as in [49], we can show that $\mathbb{E}(z^{W_n})$ is connected to Legendre polynomials by the relation

$$\begin{aligned} \mathbb{E}(z^{W_n}) &= [v^n] \frac{1}{\sqrt{(1-v)(1-zv)}} \\ &= z^{n/2} \text{Legendre}_n \left(\frac{z^{1/2} + z^{-1/2}}{2} \right), \end{aligned}$$

so that the root-unity of the left-hand side follows from the property that Legendre polynomials have only real roots over the interval $[-1, 1]$. Note that the moment generating function of the arcsin law with zero mean and unit variance is given by the Bessel function

$$\begin{aligned} \mathbb{E}(e^{(W-1/2)s/\sqrt{2}}) &= e^{-\sqrt{2}s} \left(1 + \sum_{k \geq 1} \binom{2k}{k} \frac{(s/\sqrt{2})^k}{k!} \right) \\ &= J_0(\sqrt{2}is) = \prod_{j \geq 1} \left(1 + \frac{2s^2}{\zeta_{0,j}^2} \right), \end{aligned}$$

where the $\zeta_{0,j}$'s are the positive zeros of $J_0(z)$. So we have (5) with $q = 0$ and $q_j = 4\zeta_{0,j}^{-2}$.

In a more general manner, from the Gegenbauer polynomials, one can also define the random variables W_n by

$$\begin{aligned} \mathbb{E}(z^{W_n}) &= \frac{1}{\binom{2\alpha+n-1}{n}} [v^n] \frac{1}{(1-v)^\alpha (1-zv)^\alpha} \\ &= \sum_{0 \leq j \leq n} \frac{\binom{\alpha+j-1}{j} \binom{\alpha+n-j-1}{n-j}}{\binom{2\alpha+n-1}{n}} z^j \quad (\alpha > 0), \end{aligned}$$

for which all coefficients are positive and $\mathbb{E}(z^{W_n})$ has only unit roots. The limit law W_α can be derived as in the bounded case of the Turán-Fejér polynomials

$$\begin{aligned}\mathbb{E}(e^{(W_\alpha-1/2)s}) &= \left(\frac{is}{4}\right)^{-\alpha+1/2} \Gamma(\alpha+1/2) J_{\alpha-1/2}(is/2) \\ &= \prod_{j \geq 1} \left(1 + \frac{s^2}{4\xi_{\alpha+1/2,j}^2}\right).\end{aligned}$$

Note that the random variable $2\sqrt{2\alpha+1}(W_\alpha-1/2)$ has variance one and kurtosis $3-6/(2\alpha+3)$, which lies between 1 and 3 for $\alpha > 0$.

For more potential examples, see [24, Chapter 6]. See also [10] for many many polynomials of the form

$$\mathbb{E}(z^{Y_n}) = \sum_{0 \leq j < r} p_j \sum_{j \leq k < n} \frac{\binom{k}{j} \binom{n-1-k}{r-1-j}}{\binom{n}{r}} z^k,$$

for a given probability distribution $\sum_{0 \leq j < r} p_j = 1$.

5.3 Uniform distribution

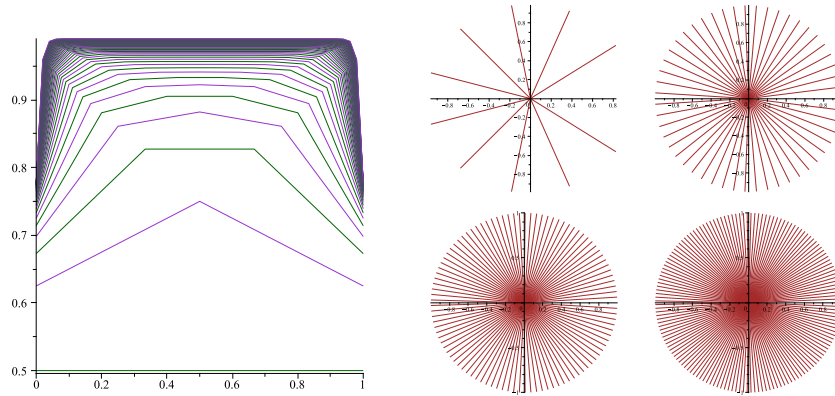


Figure 7: Normalized histograms (left) of $P_n(z)$ (scaled by standard variation) for $n = 1, \dots, 50$ and the distributions of the zeros (right) for $n = 10, 50, 100, 150$.

The literature abounds with criteria for the root-unity of polynomials. Among these, [27] proved that a complex polynomial $P(z) := \sum_{0 \leq k \leq n} a_k z^k$ with $a_k = a_{n-k}$ is root-unity if

$$|a_n| \geq \sum_{0 \leq j \leq n} |a_n - a_j|;$$

see also [43]. In particular, if the coefficients of $P(z)$ are close to a constant, then all its roots lie on the unit circle. For example, let $E_j = j![z^j](\cosh(z))^{-1}$ denote Euler's numbers; then the polynomial

$$P_n(z) = (-1)^n \sum_{0 \leq j \leq n} \binom{2n}{2j} E_{2j} E_{2n-2j} z^j = [w^n] \frac{1}{\cos(\sqrt{w}) \cos(\sqrt{wz})},$$

is root-unitary (see [28]) with non-negative coefficients. See also [29] for more information and other root-unitary polynomials. Observe that

$$\frac{(-1)^n}{(2n)!} \binom{2n}{2j} E_{2j} E_{2n-2j} \sim \frac{4^{n+2}}{\pi^{2n+2}},$$

as $j, n-j \rightarrow \infty$. Thus we can show that the random variables associated with the coefficients of $P_n(z)$ will be close to uniform, and the limit law is also uniform. Details are omitted here.

Acknowledgements

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Appendix. Approximate zeros of the limiting moment generating function arising from Reimer’s polynomials

Let $f_0(s)$ denote the moment generating function of the limit (centered) random variable X (in Theorem 5.3, Section 5.1)

$$f_0(s) := \mathbb{E}\left(e^{(X-1/2)s}\right) = 12 \sum_{m \geq 0} \frac{s^m}{m!} \sum_{0 \leq j \leq m} \binom{m}{j} (-1)^j A_{j+2} 2^{-m+j},$$

where the sequence A_k is defined in Section 5.1. Since this function contains only even powers in its Taylor expansion, we define

$$f(s) := f_0(\sqrt{s}) = \sum_{m \geq 0} \beta_m s^m = \prod_{k \geq 1} \left(1 + \frac{s}{\alpha_k}\right), \quad (36)$$

where $\alpha_1 < \alpha_2 < \dots$ is an increasing sequence and

$$\beta_m := \frac{\mathbb{E}(X - 1/2)^{2m}}{(2m)!} = 12 \sum_{0 \leq j \leq 2m} \frac{(-1)^j A_{j+2}}{j!(2m-j)!2^{2m-j}}.$$

To obtain approximate numerical values of α_k (see Watson’s classical treatise on Bessel functions [51, §15.5]), we start from computing the sequences σ_m , which is easily achieved by the recurrence (by using (36))

$$\sigma_m = (-1)^{m-1} m \beta_m - \sum_{1 \leq \ell < m} (-1)^\ell \beta_\ell \sigma_{m-\ell}.$$

From this relation, we can compute successively the values of σ_m . Euler’s idea is based on the inequalities

$$\sigma_m > \alpha_1^{-m} \quad \text{and} \quad \sigma_{m+1} < \frac{\sigma_m}{\alpha_1},$$

or

$$\sigma_m^{-1/m} < \alpha_1 < \frac{\sigma_m}{\sigma_{m+1}}.$$

We then obtain the table on the right-hand side. More calculations lead to

$$\alpha_1 \approx 53.9107695922601406201974030 \dots$$

m	$\sigma_m^{-1/m}$	σ_m/σ_{m+1}
1	30	86.8966
2	51.0580	58.4874
3	53.4231	54.9899
4	53.8107	54.1995
5	53.8883	53.9917
6	53.9055	53.9339
7	53.9093	53.9174
8	53.9104	53.9127
9	53.91068	53.9113
10	53.91075	53.9109

Once this value is determined, we proceed in a similar way by using the relations

$$\left(\sigma_m - \sum_{1 \leq j < k} \alpha_j^{-m}\right)^{-1/m} < \alpha_k < \frac{\sigma_m - \sum_{1 \leq j < k} \alpha_j^{-m}}{\sigma_{m+1} - \sum_{1 \leq j < k} \alpha_j^{-m-1}},$$

for $k = 1, 2, \dots$. We then obtain

$$\alpha_2 \approx 185.8453702068 \dots$$

$$\alpha_3 \approx 396.0710121154 \dots$$

$$\alpha_4 \approx 684.8094715040 \dots$$

All numbers here are highly sensitive to numerical errors because σ_m decreases very fast.