A binomial splitting process in connection with corner parking problems

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Abstract

A special type of binomial splitting process is studied. Such a process can be used to model a high-dimensional corner parking problem, as well as the depth of random PATRICIA tries (a special class of digital tree data structures). The later also has natural interpretations in terms of distinct values in iid geometric random variables and the occupancy problem in urn models. The corresponding distribution is marked by logarithmic mean and bounded variance, which is oscillating, if the binomial parameter p is not equal to $1/2$, and asymptotic to 1 in the unbiased case. Also, the limiting distribution does not exist owing to periodic fluctuations.

Key words. Binomial distribution, parking problem, periodic fluctuation, asymptotic approximation, digital trees, de-Poissonization.

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Introduction. We study in this paper the random variables X_n defined recursively by

$$X_n \stackrel{d}{=} X_{I_n} + 1, \qquad \text{for } n \geqslant 1, \tag{1}$$

with $X_0 = 0$, where (X_n) and (I_n) are independent and

$$\mathbb{P}(I_n = k) = \binom{n}{k} \frac{p^k q^{n-k} - p^k (q-p)^{n-k}}{1 - q^n}, \qquad \text{for } k = 0, \dots, n-1,$$

where, throughout this paper, $0 < p \leqslant q := 1 - p$. In particular,

$$p = \tfrac{1}{2} \Rightarrow \mathbb{P}(I_n = k) = \binom{n}{k} \frac{1}{2^n - 1},$$
$$p = \tfrac{1}{3} \Rightarrow \mathbb{P}(I_n = k) = \binom{n}{k} \frac{2^{n-k} - 1}{3^n - 2^n},$$

for $k = 0, \dots, n-1$. Note that, for convenience we retain the case $k = 0$, but drop $k = n$.

The random variables X_n originally arose from the analysis of a special type of parking problem with "corner preference" (described below). They would also arise in a leader election algorithm that advances a truncated binomial number of contestants at each stage. Namely, X_n is the number of rounds till the election comes to an end; see [15, 16] for a broad framework for these types of problems.

On the other hand, it turns out that, when $p = 1/2$, the distribution of X_n is identical to that of two parameters in a random symmetric PATRICIA trie: the depth (distance between a uniformly chosen leaf node and the root) and the length of the "left arm" (the path that starts at the root and keeps going left, until no more nodes can be found on the left); see for example [17, 26]. Also, the depth or the left arm of random PATRICIA tries is identically distributed as the number of distinct values in some random sequences (see [1]), and the number of occupied urns in some urn models (see [9]).

We prove in this paper that the random variables X_n have logarithmic mean and bounded variance for large n. Also the distributions do not approach a fixed limit law due to the inherent fluctuations. For a similar context, see [13, 14] and the references therein; see also the recent paper [16].

A corner preference parking problem in discrete space. The parking problem has a long history in the discrete probability literature, and is closely connected to many applications and models in chemistry, physics, biology and computer algorithms; see [2, 3]. Most analytic results for the numerous variants in the literature have to do with one-dimensional settings, and very few deal with higher dimensions due to the intrinsic complexity of the corresponding equations.

We first explain a simple discrete parking problem. Integral translates of the cube $[0, \ell]^n$ are "parked" into the n-dimensional hypercube $[0, L]^n$, where $L > \ell \geqslant 1$. A precise mathematical formulation of this is as follows: represent cubes by their corner which has the shortest distance to the origin. Moreover, set

$$Z^n_{L-\ell} := \{\mathbf{a} = (a_1, \dots, a_n) : a_j = 0, 1, \dots, L - \ell, \text{for } 1 \leqslant j \leqslant n\},$$

and define a distance $\rho(\mathbf{x},\mathbf{y}) := \max_{1\leqslant j\leqslant n}|x_j - y_j|$ between two points $\mathbf{x},\mathbf{y} \in Z^n_{L-\ell}$. At first, choose one point uniformly at random from $Z^n_{L-\ell}$ and record it as $\mathbf{a}(1)$. Then choose another point uniformly at random and record it as $\mathbf{a}(2)$ if $\ell \leqslant \rho(\mathbf{a}(1),\mathbf{a}(2))$; otherwise, reject it and repeat the same procedure. If $\mathbf{a}(1),\mathbf{a}(2),\ldots,\mathbf{a}(k)$ are already recorded, choose the next point uniformly at random and record it as $\mathbf{a}(k+1)$ if $\ell \leqslant \rho(\mathbf{a}(k+1),\mathbf{a}(j))$ for $j = 1,2,\ldots,k$; otherwise, reject it and repeat the same procedure. We continue this procedure until it is impossible to add more points among the $(L-\ell+1)^n$ points. Since analytic development of this model remains challenging, simulations have been carried out for finding the jamming density of this model; see [10, 11].

We further restrict the parking to be operated along one direction only, which we call "corner preference parking." More precisely, let

$$S(\mathbf{a}) := \{\mathbf{x} : \mathbf{x} \in Z^n_{L-\ell}, 0 \leqslant x_j \leqslant a_j, j = 1,\ldots,n\},$$
$$U(\mathbf{a},\ell) := \{\mathbf{x} : \mathbf{x} \in Z^n_{L-\ell}, \rho(\mathbf{x},\mathbf{a}) < \ell\},$$

and $S_U(\mathbf{a},\ell) := S(\mathbf{a}) \setminus U(\mathbf{a},\ell)$.

The corner preference parking problem then starts from $S_U(\mathbf{a(1)},\ell)$ with $\mathbf{a(1)} = (L-\ell, L-\ell, ..., L-\ell) \in Z^n_{L-\ell}$. After that, we place sequentially at random the integral translates of cube $[0,\ell]^n$ into the cube $[0,L]^n$, so that any car placed is closer to the "corner" (origin) than the previously placed cubes, until there is no possible space to park. By "closer to the corner" we mean that the coordinates of the point representing the car are all at most as large as the car parked immediately before it. The process continues till saturation.

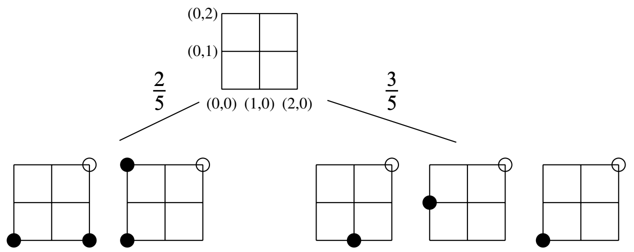

Figure 1: *The five* (3^2-2^2) *different two-dimensional configurations of corner preference parking, when* $L = 4$ *and* $\ell = 2$*. In this case* $\mathbb{E}(u^{X_2}) = \frac{3}{5}u + \frac{2}{5}u^2$.

Take now $L = 2m$, and $\ell = m$, where $m \geqslant 1$. Assume that all possible "parking positions" are equally likely at each stage. Let the random variable Y_n be the number of cars parked after the first car at the time of saturation in such an n–dimensional corner parking problem. The distribution of Y_n can be explicitly characterized.

Lemma 1. *The random variables* Y_n *can be recursively enumerated by*

$$\mathbb{E}(u^{Y_n}) = u \sum_{1\leqslant k\leqslant n} \binom{n}{k} \frac{m^k-(m-1)^k}{(m+1)^n - m^n}\, \mathbb{E}\big(u^{Y_{n-k}}\big), \qquad \textit{for } n \geqslant 1, \tag{2}$$

with $Y_0 = 0$.

This corresponds to (1) with $p = 1/(m+1)$.

Proof. To prove (2), consider the special case, when $L = 4$ and $\ell = 2$. Let $T(n,k)$ be the set of all vectors $\mathbf{x} \in Z_2^n$, such that k elements are 0 or 1 (and with at least one 0) and the $n-k$ remaining elements are 2. The cardinality of this set $T(n,k)$ is $\binom{n}{k}(2^k-1)$, and each of these elements is chosen to be $\mathbf{a}(2)$ with the same probability $1/(3^n-2^n)$. Since $\bigcup_{0\leqslant k<n} T(n,k) = S_U(\mathbf{a}(1),2)$, we obtain (2), when $L=4$ and $\ell=2$; see Figure fig-2d for an illustration. The proof for the general case is similar and omitted. □

Depths of PATRICIA tries. Tries (a mixture of tree and re*trie*val) are one of the most useful tree structures in storing alphabetical or digital data in computer algorithms, the underlying construction principle of which being simply "0-bit directing to the left" and "1-bit directing to the right". PATRICIA[1] tries are a variant of tries where all nodes with only a single child are compressed; see Figure 2 for a plot of tries and PATRICIA tries and the book [21] for more information. Note that, unlike tries whose number of internal nodes is not necessarily a constant, a PATRICIA trie of n keys has always $n-1$ internal nodes for branching purposes, a standard property of a tree.

To study the shapes of random PATRICIA tries, we assume that the input is a sequence of n independent and identically distributed random variables, each composed of an infinite sequence of Bernoulli random variables with mean p, $0<p<1$. Under such a Bernoulli model, we construct random PATRICIA tries and the shape parameters become random variables.

Consider the depth Z_n of a random PATRICIA trie of n keys under the Bernoulli model, where the depth denotes the distance between the root and a randomly chosen key (from the leaves where keys are stored), where the n keys are equally likely to be selected. Then we have the recurrence relation for the probability generating function of Z_n

$$\mathbb{E}(u^{Z_n}) = u \sum_{1\leqslant k<n} \frac{\binom{n}{k} p^k q^{n-k}}{1-p^n-q^n} \left(\frac{k}{n}\,\mathbb{E}(u^{Z_k}) + \frac{n-k}{n}\,\mathbb{E}(u^{Z_{n-k}})\right), \qquad \text{for } n \geqslant 2,$$

with $Z_0 = Z_1 = 0$. In the unbiased case $p = 1/2$, this reduces to

$$\mathbb{E}(u^{Z_n}) = u \sum_{0\leqslant k\leqslant n-2} \frac{\binom{n-1}{k}}{2^{n-1}-1}\,\mathbb{E}(u^{Z_{k+1}}),$$

which implies that

$$Z_{n+1} \stackrel{d}{\equiv} X_n, \qquad \text{for } n \geqslant 1; \quad p = \frac{1}{2}.$$

Another identically distributed random variable is the length W_n of the "left arm", which is the path starting from the root and going always to the left until reaching a key-node. Then, under the Bernoulli model,

$$\mathbb{E}(u^{W_n}) = u \sum_{1\leqslant k\leqslant n} \frac{\binom{n}{k} p^k q^{n-k}}{1-q^n}\,\mathbb{E}(u^{W_{n-k}}), \qquad \text{for } n \geqslant 2, \tag{3}$$

with $W_0 = 0$ and $W_1 = 1$. It is obvious that $W_n \stackrel{d}{\equiv} X_n$, when $p = 1/2$.

[1] "PATRICIA" is an acronym, which stands for "Practical Algorithm To Retrieve Information Coded In Alphanumeric".

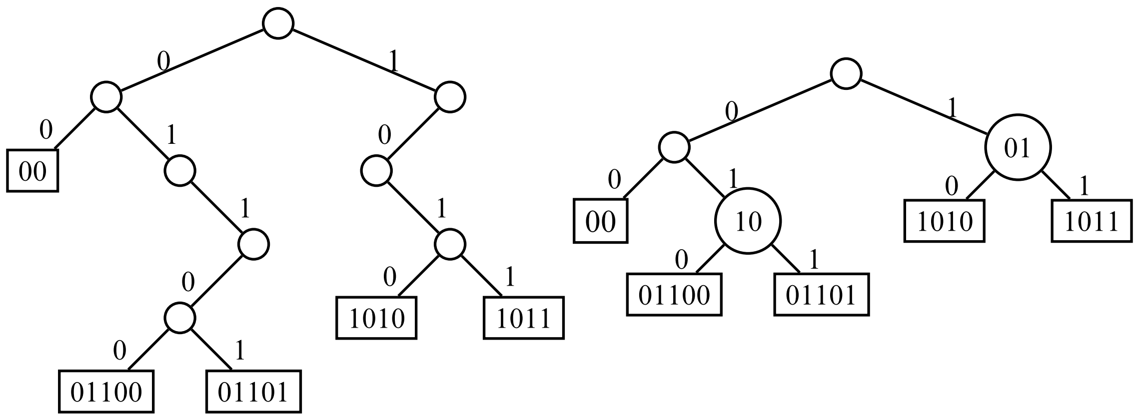

Figure 2: *A trie (left) of $n = 5$ records, and the corresponding PATRICIA tries (right): the circles represent internal nodes and rectangles holding the records are external nodes. The compressed bits are also indicated on the nodes.*

Distinct values and urn models. The left arm W_n has yet two other different interpretations. One is in terms of the number of distinct letters in a sequence of independent and identically distributed geometric random variables with success probability p for which one has exactly the recurrence (3); see [1]. Alternatively, if we consider the urn model where the jth urn has probability of pq^j of receiving a ball, then the number of occupied urns also follows the same distribution; see [9].

Exponential generating functions. The exponential generating function for the moment generating function of X_n,

$$P(z, y) := \sum_{n \geqslant 0} \frac{\mathbb{E}(e^{X_n y})}{n!} z^n,$$

satisfies, by (1), the functional equation

$$P(z, y) = e^y (e^{qz} - e^{(q-p)z}) P(pz, y) + P(qz, y).$$

It follows that the exponential generating function of the mean $f_1(z) := \sum_{n \geqslant 0} \mathbb{E}(X_n) z^n / n!$ satisfies

$$f_1(z) = (e^{qz} - e^{(q-p)z}) f_1(pz) + f_1(qz) + e^z - e^{qz},$$

with $f_1(0) = 0$. By iteration, we obtain

$$f_1(z) = \sum_{\substack{k \geqslant 0 \\ 0 \leqslant j \leqslant k}} \left(e^{p^{k-j} q^j z} - e^{p^{k-j} q^{j+1} z} \right) \\ \times \sum_{0 \leqslant i_1 \leqslant \cdots \leqslant i_{k-j} \leqslant j} \prod_{0 \leqslant \ell < k-j} \left(e^{p^\ell q^{i_{\ell+1}+1} z} - e^{(q-p) p^\ell q^{i_{\ell+1}} z} \right),$$

which does not seem useful for further manipulation.

Poisson generating functions. For our asymptotic purposes, it is technically more convenient to consider the Poisson generating function,

$$\tilde{P}(z,y) := e^{-z} P(z,y),$$

which then satisfies the equation

$$\tilde{P}(z,y) = e^{y}(1-e^{-pz})\tilde{P}(pz,y) + e^{-pz}\tilde{P}(qz,y). \tag{4}$$

It follows that the Poisson generating function for the mth moment

$$\tilde{f}_m(z) := e^{-z}\sum_{n\geqslant 0}\frac{\mathbb{E}(X_n^m)}{n!}z^n,$$

satisfies the equation

$$\tilde{f}_m(z) = (1-e^{-pz})\sum_{0\leqslant\ell\leqslant m}\binom{m}{\ell}\tilde{f}_\ell(pz) + e^{-pz}\tilde{f}_m(qz), \qquad \text{for } m\geqslant 0,$$

where $\tilde{f}_0(z) = 1$.

In particular, we have

$$\tilde{f}_1(z) = (1-e^{-pz})\tilde{f}_1(pz) + e^{-pz}\tilde{f}_1(qz) + 1 - e^{-pz}, \tag{5}$$

$$\tilde{f}_2(z) = (1-e^{-pz})\tilde{f}_2(pz) + e^{-pz}\tilde{f}_2(qz) + 2(1-e^{-pz})\tilde{f}_1(pz) + 1 - e^{-pz}. \tag{6}$$

Expected value of X_n. Let

$$\phi(z) = e^{-z}\left(\tilde{f}_1\left(p^{-1}qz\right) - \tilde{f}_1(z)\right), \tag{7}$$

and let $\phi^*(s)$ denote its Mellin transform (see [4])

$$\phi^*(s) := \int_0^\infty e^{-t}t^{s-1}\left(\tilde{f}_1\left(p^{-1}qt\right) - \tilde{f}_1(t)\right)\,\mathrm{d}t, \tag{8}$$

which is well-defined in the half-plane $\Re(s) > -1$ (see Appendix for growth properties of $\tilde{f}_1$).

Theorem 1. *The expected value of X_n satisfies*

$$\mathbb{E}(X_n) = \log_{1/p} n + \frac{\gamma+\phi^*(0)}{\log(1/p)} - \frac{1}{2} + Q(\log_{1/p} n) + O(n^{-1}).$$

Here γ denotes Euler's constant and

$$Q(u) := \sum_{k\in\mathbb{Z}\setminus\{0\}} Q_k e^{-2k\pi i u}, \qquad Q_k := -\frac{\Gamma(\chi_k) - \phi^*(\chi_k)}{\log(1/p)}, \tag{9}$$

where $\chi_k := 2k\pi i/\log(1/p)$, and Γ denotes the Gamma function.

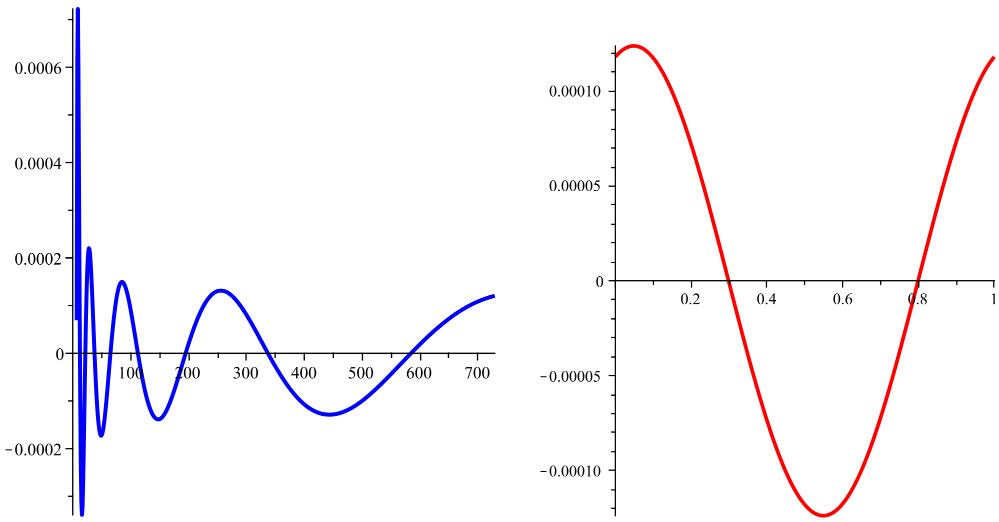

Figure 3: $p = 1/3$: *Fluctuation of the periodic function $Q(\log_3 n)$, as approximated by $\mu_n - H_n/\log 3 + 1/2 - \phi^*(0)/\log 3$ (left) and the first five terms of the Fourier series (9) (right).*

The asymptotic expansion simplifies when $p = 1/2$; indeed, in this case, we have the closed-form expression

$$\tilde{f}_1(z) = \sum_{k\geqslant 1}\left(1 - e^{-z/2^k}\right), \qquad \text{for } \Re(z) > 0.$$

Corollary 1. *In the symmetric case when $p = 1/2$, the expected value of X_n satisfies asymptotically*

$$\mathbb{E}(X_n) = \log_2 n + \frac{\gamma}{\log 2} - \frac{1}{2} - \frac{1}{\log 2}\sum_{k\neq 0}\Gamma(\chi_k)n^{-\chi_k} + O\left(n^{-1}\right),$$

where $\chi_k = 2k\pi i/\log 2$.

For numerical purposes, the value of $\phi^*(\chi_k)$ can be computed by the series expression

$$\phi^*(\chi_k) = \sum_{j\geqslant 1}\frac{\mu_j}{j!}\Gamma(\chi_k + j)\left(q^j - 2^{-j-\chi_k}\right), \qquad \text{for } k = 0, 1, \ldots .$$

Approximate plots of the periodic function $Q(u)$ for $p = 1/3$ based on exact values of μ_n and on its Fourier series are given in Figure 3.

Outline of proof. Theorem 1 is proved by a two-stage, purely analytic approach based on Mellin transform and analytic de-Poissonization (see [5, 12]). We outline the major steps and arguments used here, leaving the major technical justification in the Appendix.

Our starting point is the functional equation (5), which is rewritten as

$$\tilde{f}_1(z) = \tilde{f}_1(pz) + \phi(pz) + 1 - e^{-pz},$$

where $\phi(z)$ is defined in (7). While ϕ involves itself $\tilde{f}$, we show that it is exponentially small for large complex parameter, and thus the asymptotics of $\tilde{f}_1(z)$ can be readily derived by standard inverse Mellin transform arguments (growth order of the integrand at infinity and calculus of residues).

Once the asymptotics of $\tilde{f}_1(z)$ for large $|z|$ is known, we can apply the Cauchy integral formula

$$\mathbb{E}(X_n) = \frac{n!}{2\pi i} \oint_{|z|=n} z^{-n-1} e^z \tilde{f}_1(z)\, \mathrm{d}z,$$

and the saddle-point method to derive the asymptotics of the mean. Roughly, the growth order of $\tilde{f}_1$ is small, meaning that the saddle-point (where the derivative of the integrand becomes zero) lies near n. The specialization of the saddle-point method here (with integration contour $|z| = n$) has many interesting properties and is often referred to as the analytic de-Poissonization (see the survey paper by Jacquet and Szpankowski [12]).

It turns out that such a Mellin and de-Poissonization process can be manipulated in a rather systematic and operational manner by introducing the notion of JS-admissible functions in which we combine ideas from [12] and [7] (see also [6, 8]). So we can easily apply the same approach to characterize the asymptotics of the variance and the limiting distribution.

Mellin transform. Let

$$\mathscr{S}_\varepsilon := \{z \,:\, |\arg(z)| \leqslant \pi/2 - \varepsilon\}, \qquad \text{for } \varepsilon > 0. \tag{10}$$

By Proposition 3 (in Appendix), $\tilde{f}_1(z)$ is polynomially bounded for large $|z|$ in the sector $\mathscr{S}_\varepsilon$. This means that $\phi(z) + 1 - e^{-pz} = O(1)$ for $|z| \geqslant 1$ in $\mathscr{S}_\varepsilon$. Consequently, $\tilde{f}_1(z) = O(|\log z|)$ in the same range of z. On the other hand, since $\tilde{f}_1(z) \sim z$, as $z \to 0$, we see that the Mellin transform

$$\tilde{f}_1^\star(s) := \int_0^\infty \tilde{f}_1(z) z^{s-1}\, \mathrm{d}z,$$

exists in the strip $-1 < \Re(s) < 0$, and defines an analytic function there.

It follows from (5) that

$$\tilde{f}_1^\star(s) = \frac{\Gamma(s) - \phi^*(s)}{1 - p^s}, \qquad \text{for } \; -1 < \Re(s) < 0,$$

and ϕ^* is defined in (8).

By Mellin inversion formula,

$$\tilde{f}_1(z) = \frac{1}{2\pi i} \int_{-1/2-i\infty}^{-1/2+i\infty} \frac{\Gamma(s) - \phi^*(s)}{1 - p^s}\, z^{-s}\, \mathrm{d}s.$$

We need the growth property of $\phi^*(s)$ for $s = \sigma \pm i\infty$.

Lemma 2. *For $\sigma > -1$,*

$$|\phi^*(\sigma \pm it)| = O(e^{-(\pi/2-\varepsilon)|t|}),$$

as $|t| \to \infty$.

Proof. This follows from the fact that $\tilde{f}_1(z)$ is an entire function, the estimate $\tilde{f}_1(z) = O(|\log z|)$ for $z \in \mathscr{S}_\varepsilon$ and the Exponential Smallness Lemma ([4, Proposition 5]). □

On the other hand, since

$$|\Gamma(\sigma \pm it)| = O\left(|t|^{\sigma-1/2}e^{-\pi|t|/2}\right),$$

for finite σ and $|t| \to \infty$, we can move the line of integration to the right, summing the residues of all poles encountered. The result is

$$\tilde{f}_1(z) = \log_{1/p} z + C + Q(\log_{1/p} z) + \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \frac{\Gamma(s) - \phi^*(s)}{1-p^s}\, z^{-s}\,\mathrm{d}s, \qquad (11)$$

where (defining $\chi_k := 2k\pi i/\log(1/p)$)

$$C := -\frac{1}{2} + \frac{\gamma + \phi^*(0)}{\log(1/p)}. \qquad (12)$$

Note that, by definition, we have

$$\phi^*(\chi_k) = \sum_{j\geqslant 1} \frac{\mu_j}{j!}\,\Gamma(j+\chi_k)\left(q^j - 2^{-j-\chi_k}\right), \qquad \text{for } k \in \mathbb{Z},$$

the series being absolutely convergent by the growth order of μ_j. In particular, when $p = 1/3$

$$\phi^*(0) = \sum_{j\geqslant 1} \frac{\mu_j}{j}\left(\frac{2^j}{3^j} - \frac{1}{2^j}\right) \approx 0.58130\,98083\,52813\,44019\ldots,$$

so that $C \approx 0.55453\,53308\,02526\,96605\ldots$.

To evaluate the remainder integral in (11), we expand the factor $1/(1-p^s)$ (since $\Re(s) > 0$) in geometric series, and integrate term by term, giving

$$\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \frac{\Gamma(s) - \phi^*(s)}{1-p^s}\, z^{-s}\,\mathrm{d}s = \sum_{k\geqslant 0} e^{-p^{-k}z}\left(1 - \tilde{f}_1(qp^{-k-1}z) + \tilde{f}_1(p^{-k}z)\right).$$

Thus the remainder is indeed exponentially small.

We summarize these derivations as follows.

Proposition 1. *For z lying in the sector $\mathscr{S}_\varepsilon$, $\tilde{f}_1(z)$ satisfies the asymptotic and exact formula*

$$\tilde{f}_1(z) = \log_{1/p} z + C + Q(\log_{1/p} z) + \sum_{k\geqslant 0} e^{-p^{-k}z}\left(1 - \tilde{f}_1(qp^{-k-1}z) + \tilde{f}_1(p^{-k}z)\right), \qquad (13)$$

where C and Q are given in (12) and (9), respectively.

Theorem 1 then follows from standard de-Poissonization argument (see Appendix)

$$\mathbb{E}(X_n) = \tilde{f}_1(n) - \frac{n}{2}\tilde{f}_1''(n) + O\left(n^{-2}\right),$$

and (13).

The variance. For the asymptotics of the variance, it proves advantageous to consider suitable Poissonized variance at the generating function level, which, in the case of X_n, can be handled by the following form

$$\tilde{V}(z) := \tilde{f}_2(z) - \tilde{f}_1(z)^2,$$

where the Poisson generating function of the second moment $\tilde{f}_2(z)$ satisfies the equation (6). Then $\tilde{V}(z)$ satisfies the functional equation

$$\tilde{V}(z) = (1 - e^{-pz})\tilde{V}(pz) + e^{-pz}\tilde{V}(qz) + g_V(z), \tag{14}$$

with $\tilde{V}(0) = 0$, where

$$g_V(z) := e^{-pz}\left(1 - e^{-pz}\right)\left(1 + \tilde{f}_1(pz) - \tilde{f}_1(qz)\right)^2.$$

Unlike g and g_2, which is $O(1)$ for large z, g_V is exponentially small for large z.

When $p = 1/2$, we see that (14) has the closed-form solution

$$\tilde{V}(z) = 1 - e^{-z}.$$

When $p \neq q$, define

$$\phi_V^*(s) := \int_0^\infty e^{-z}z^{s-1}\left(\tilde{V}\left(p^{-1}qz\right) - \tilde{V}(z) + (1 - e^{-z})\left(1 + \tilde{f}_1(z) - \tilde{f}_1\left(p^{-1}qz\right)\right)^2\right) \mathrm{d}z.$$

By following exactly the same analysis as that for $\tilde{f}_1$, we obtain

$$\begin{aligned}\tilde{V}(z) = Q_V(\log_{1/p} z) + \sum_{k\geqslant 0} e^{-p^{-k}z}\Big\{&\tilde{V}\left(p^{-k-1}qz\right) - \tilde{V}(p^{-k}z) \\ &+ \left(1 - e^{-p^{-k}z}\right)\left(1 + \tilde{f}_1(p^{-k}z) - \tilde{f}_1\left(p^{-k-1}qz\right)\right)^2\Big\},\end{aligned} \tag{15}$$

for $\Re(z) > 0$, where

$$Q_V(u) = \frac{1}{\log(1/p)}\sum_{k\in\mathbb{Z}}\phi_V^*(\chi_k)e^{-2k\pi iu}.$$

Note that $Q_V(u) = 1$ when $p = 1/2$.

Theorem 2. *If $p = 1/2$, then the variance of X_n satisfies*

$$\mathbb{V}(X_n) = 1 + O(n^{-1});$$

if $p \neq q$, then the variance of X_n is bounded and asymptotically periodic in nature

$$\mathbb{V}(X_n) = Q_V(\log_{1/p} n) + O(n^{-1}).$$

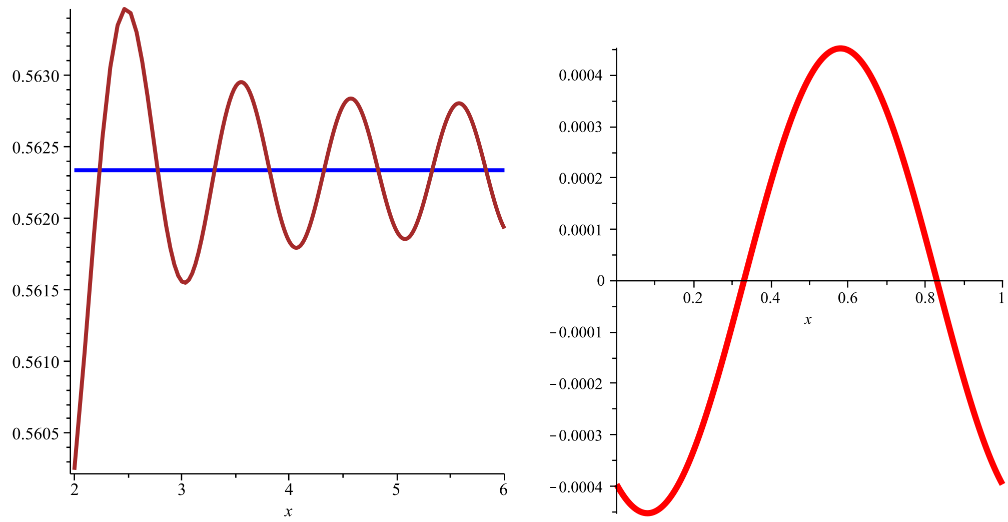

Figure 4: $p = 1/3$*: Fluctuation of the periodic function* $Q_V(\log_3 n)$*, as approximated by* $\mathbb{V}(X_n) + c_0/n - c_0/(2n^2)$ *in logarithmic scale (left) and the first four oscillating terms of its Fourier series (right). Here the number* $c_0 = 1/\log(1/p)^2$ *and the two additional terms* $c_0/n - c_0/(2n^2)$ *are chosen for a better numerical correction and graphical display.*

Proof. By the definition of $\tilde{V}$

$$\begin{aligned}\mathbb{V}(X_n) &= n![z^n]e^z \tilde{f}_2(z) - (\mathbb{E}(X_n))^2 \\ &= n![z^n]e^z \tilde{V}(z) - n![z^n]e^z \tilde{f}_1(z)^2 - \left(n![z^n]e^z \tilde{f}_1(z)\right)^2,\end{aligned}$$

which, by the asymptotic nature of the Poisson-Charlier expansions (see Appendix), is asymptotic to

$$\begin{aligned}\mathbb{V}(X_n) &= \tilde{V}(n) + O\left(n\tilde{V}''(n) + n\tilde{f}_1'(n)^2\right) \\ &= \tilde{V}(n) + O\left(n^{-1}\right),\end{aligned}$$

and the theorem follows from (15). □

Figure 4 illustrates the periodic fluctuations of the variance when $p = 1/3$. See also [24] for a similar situation where the variance is not oscillating when $p = 1/2$.

For computational purpose, we use the series expression

$$\begin{aligned}\phi_V^*(\chi_k) &= \sum_{j\geqslant 1}\frac{\mathbb{E}(X_j^2)}{j!}\Gamma(j+\chi_k)\left(q^j-2^{-j-\chi_k}\right)+\Gamma(\chi_k)\left(1-2^{-\chi_k}\right)\\ &\quad+\sum_{j\geqslant 1}\frac{\mu_j^{[2]}}{j!}\Gamma(j+\chi_k)\left(2\cdot 3^{-j-\chi_k}-4^{-j-\chi_k}-q^j2^{-j-\chi_k}\right)\\ &\quad+2\sum_{j\geqslant 1}\frac{\mu_j}{j!}\Gamma(j+\chi_k)\left(2^{-j-\chi_k}-3^{-j-\chi_k}-q^j+q^j(1+p)^{-j-\chi_k}\right)\\ &\quad-2\sum_{j\geqslant 1}\frac{\mu_j^{[11]}}{j!}\Gamma(j+\chi_k)\left((1+p)^{-j-\chi_k}-(1+2p)^{-j-\chi_k}\right),\end{aligned}$$

where $\mu_n^{[2]} := n![z^n]f_1(z)^2$ and $\mu_n^{[11]} := n![z^n]f_1(pz)f_1(qz)$.

Asymptotic distribution. We now show that the distribution of X_n is asymptotically fluctuating and no convergence to a fixed limit law is possible. We focus on deriving an asymptotic approximation to the probability $\mathbb{P}(X_n = k)$, which then leads to an effective estimate for the corresponding distribution functions. The method of proof we use here relies on the same analytic de-Poissonization procedure we used for the first two moments, and requires a uniform estimate with respect to k; see [14, 20] for a similar analysis.

We begin by considering

$$\tilde{A}_k(z) = e^{-z}\sum_{n\geqslant 0}\mathbb{P}(X_n = k)\frac{z^n}{n!},$$

which satisfies the obvious bound $\tilde{A}_k(x) \leqslant 1$ for real $x \geqslant 0$. On the other hand, from (4), it follows that

$$\begin{aligned}\tilde{A}_0(z) &= e^{-z}\\ \tilde{A}_{k+1}(z) &= (1-e^{-pz})\,\tilde{A}_k(pz)+e^{-pz}\tilde{A}_{k+1}(qz)\quad (k\geqslant 0).\end{aligned}\tag{16}$$

Iterating (16) gives

$$\tilde{A}_{k+1}(z) = \sum_{j\geqslant 0}e^{-(1-q^j)z}\left(1-e^{-pq^jz}\right)\tilde{A}_k(pq^jz),\qquad \text{for } k\geqslant 0.$$

We then deduce the explicit expressions

$$\tilde{A}_k(z) = \sum_{j_1,\dots,j_k\geqslant 0} e^{-(1-q\sum_{1\leqslant r\leqslant k}p^{r-1}q^{j_1+\cdots+j_r})z}\prod_{1\leqslant r\leqslant k}\left(1-e^{-p^rq^{j_1+\cdots+j_r}z}\right),\tag{17}$$

for $k\geqslant 1$.

Now define the normalizing function

$$\Omega(z) := \prod_{j\geqslant 0}\left(1-e^{-p^{-j}z}\right).$$

For convenience, let

$$\eta(n) := \{\log_{1/p} n\}$$

denote the fractional part of $\log_{1/p} n$.

Theorem 3. *The distribution of X_n satisfies*

$$\mathbb{P}(X_n = \lfloor \log_{1/p} n \rfloor + k) = \sum_{j \geqslant 0} \hat{R}_j \left(p^{-\eta(n)+k-j} \right) + O\left(\frac{1}{n}\right), \tag{18}$$

where $\hat{R}_k(z) := \Omega(z) e^{-pz} \tilde{A}_k(qz)$ and $\tilde{A}_k(z)$ is given in (17).

Proof. Write $\hat{A}_k(z) := \Omega(z)\tilde{A}_k(z)$. Then by (16), we have

$$\hat{A}_{k+1}(z) = \hat{A}_k(pz) + \hat{R}_{k+1}(z), \qquad \text{for } k \geqslant 0,$$

which, after iteration, leads to

$$\hat{A}_k(z) = \sum_{0 \leqslant j \leqslant k} \hat{R}_j(p^{k-j} z), \qquad \text{for } k \geqslant 0,$$

or, equivalently,

$$\tilde{A}_k(z) = \frac{1}{\Omega(z)} \sum_{0 \leqslant j \leqslant k} \hat{R}_j(p^{k-j} z), \qquad \text{for } k \geqslant 0.$$

Since, by definition

$$\mathbb{P}(X_n = k) = n! [z^n] e^z \tilde{A}_k(z),$$

we need the following uniform estimates (which are needed to justify the de-Poissonization; see Appendix).

Lemma 3. *The functions $\tilde{A}_k(z)$ are uniformly JS-admissible, namely, for $|\arg(z)| \leqslant \varepsilon, 0 < \varepsilon < \pi/2$,*

$$\tilde{A}_k(z) = O\left(|z|^{\varepsilon'}\right), \tag{19}$$

uniformly in z, and, for $\varepsilon \leqslant |\arg(z)| \leqslant \pi$,

$$e^z \tilde{A}_k(z) = O\left(e^{(1-\varepsilon')|z|}\right), \tag{20}$$

uniformly in z. Here $0 < \varepsilon' < 1$ and the involved constants in both cases are absolute.

Proof. Consider first $|\arg(z)| \leqslant \varepsilon$. Choose $K > 0$ large enough such that $1+2e^{-p\Re(z)} \leqslant 1+\varepsilon'$ for all z with $|z| > K$. Moreover, choose $C > 0$ such that for $1 \leqslant |z| \leqslant K$

$$|\tilde{A}_k(z)| \leqslant e^{|z| - \Re(z)} \leqslant C \qquad \text{for } k \geqslant 0.$$

We use a simple induction to show that

$$|\tilde{A}_k(z)| \leqslant C |z|^{\log_{1/q}(1+\varepsilon')} \qquad \text{for } k \geqslant 0. \tag{21}$$

A similar inductive proof is used in [12] where it is referred to as induction over increasing domains. The claim (21) holds for $k = 0$. Next we assume (21) has been proved for k and we prove it for $k+1$. The case $1 \leqslant |z| \leqslant K$ follows from the definition of C. If $K < |z| \leqslant K/q$, we can use (16) and the induction hypothesis, and obtain

$$\begin{aligned}|\tilde{A}_{k+1}(z)| &\leqslant \left(1 + e^{-p\Re(z)}\right)|\tilde{A}_k(pz)| + e^{-p\Re(z)}|\tilde{A}_{k+1}(qz)| \\ &\leqslant C(1+\varepsilon')|qz|^{\log_{1/q}(1+\varepsilon')} = C|z|^{\log_{1/q}(1+\varepsilon')}.\end{aligned}$$

Continuing successively the same argument with $K/q^j < |z| \leqslant K/q^{j+1}$ for $j \geqslant 1$, the upper bound (21) follows for all z. This concludes the proof of (19).

To prove (20), let $A_k(z) := e^z \tilde{A}_k(z)$. Then (16) becomes

$$A_{k+1}(z) = \left(e^{qz} - e^{(q-p)z}\right) A_k(pz) + A_{k+1}(qz) \qquad (k \geqslant 0).$$

Note that we have the (trivial) bound $|A_k(z)| \leqslant e^{|z|}$. Plugging this into the functional equation above yields

$$|A_{k+1}(z)| \leqslant \left(e^{q\cos\varepsilon|z|} + e^{(q-p)\cos(\varepsilon)|z|}\right) e^{p|z|} + e^{q|z|},$$

from which (20) follows. □

By a standard de-Poissonization argument (see Appendix for details and references), we obtain

$$\mathbb{P}(X_n = k) = \frac{1}{\Omega(n)} \sum_{0\leqslant j\leqslant k} \hat{R}_j(p^{k-j}n) + O\left(\frac{1}{n^{1-\varepsilon}}\right), \tag{22}$$

uniformly in k, where $\varepsilon > 0$ is an arbitrary small constant.

Note that we have the identity

$$\frac{1}{\Omega(n)} \sum_{k\geqslant 0} \sum_{0\leqslant j\leqslant k} \hat{R}_j(p^{k-j}n) = 1. \tag{23}$$

This is seen as follows.

$$\begin{aligned}\sum_{k\geqslant 0} \sum_{0\leqslant j\leqslant k} \hat{R}_j(p^{k-j}n) &= \sum_{j\geqslant 0} \sum_{k\geqslant 0} \hat{R}_j(p^k n) \\ &= \sum_{k\geqslant 0} \Omega(p^k n) e^{-p^{k+1}n} \sum_{j\geqslant 0} \tilde{A}_j(qp^k n) \\ &= \sum_{k\geqslant 0} \Omega(p^k n)\left(1 - \left(1 - e^{-p^{k+1}n}\right)\right) \\ &= \sum_{k\geqslant 0} \left(\Omega(p^k n) - \Omega(p^{k+1}n)\right) \\ &= \Omega(n),\end{aligned}$$

which proves (23).

Now

$$\Omega(n) = \prod_{j\geqslant 0}\left(1 - e^{-p^{-j}n}\right) = 1 + O\left(e^{-n}\right).$$

This and (22) implies that

$$\sum_{0\leqslant j\leqslant k} \hat{R}_j(p^{k-j}n) = O(1),$$

uniformly in k. Thus

$$\mathbb{P}(X_n = k) = \sum_{0\leqslant j\leqslant k} \hat{R}_j(p^{k-j}n) + O\left(\frac{1}{n^{1-\varepsilon}}\right),$$

uniformly in k.

Finally, observe that

$$\sum_{j\geqslant k+1} \hat{R}_j(p^{k-j}n) \leqslant \sum_{j\geqslant k+1} e^{-p^{k+1-j}n} = \sum_{j\geqslant 0} e^{-p^{-j}n} = O\left(e^{-n}\right).$$

Thus

$$\mathbb{P}(X_n = k) = \sum_{j\geqslant 0} \hat{R}_j(p^{k-j}n) + O\left(\frac{1}{n^{1-\varepsilon}}\right), \tag{24}$$

uniformly in k.

Since the mean is asymptotic to $\log_{1/p} n$, we replace k by $\lfloor \log_{1/p} n\rfloor + k$. Then

$$\mathbb{P}(X_n = \lfloor \log_{1/p} n\rfloor + k) = \sum_{j\geqslant 0} \hat{R}_j(p^{-\eta(n)+k-j}) + O\left(\frac{1}{n^{1-\varepsilon}}\right),$$

uniformly in k, where $\eta(n) = \{\log_{1/p} n\}$. Because of the periodicity, the limiting distribution of $X_n - \lfloor \log_{1/p} n\rfloor$, in general, does not exist. The series on the right-hand side sums (over all k) asymptotically to 1 by (23).

Finally, the finer error term $O(n^{-1})$ in (18) is obtained by refining the same procedure by including, say, one more term in the asymptotic expansion. □

On the other hand, from (17), we see that $\mathbb{P}(X_n = k)$ is exponentially small for $k = O(1)$.

A similar analysis can be given for the distribution function of X_n (one only has to divide (4) by $1 - e^y$). This then yields the following estimate for the distribution.

Corollary 2. *The distribution function of X_n satisfies*

$$\mathbb{P}(X_n - \lfloor \log_{1/p} n\rfloor \leqslant k) = \sum_{j\geqslant 0} \hat{S}_j(p^{-\eta(n)+k-j}) + O\left(\frac{1}{n}\right),$$

where $\hat{S}_k(z) = \sum_{j\leqslant k} \hat{R}_j(z)$.

When $p = 1/2$, we have the representation

$$\sum_{k\geqslant 0} \tilde{A}_k(z)u^k = \prod_{j\geqslant 1} \left(1 + (u-1)(1 - e^{-z/2^j})\right),$$

which gives

$$\tilde{A}_k(z) = e^{-z} \sum_{1\leqslant j_1<\cdots<j_k} \prod_{1\leqslant r\leqslant k} \left(e^{z/2^{j_r}} - 1\right);$$

compare with (16). This expression was already derived in [26] where different expressions of the asymptotic distributions are given.

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Appendix. Analytic de-Poissonization and JS-admissible functions

We develop required tools for justifying the growth order of the functions involved in this paper, as well as systematic means for justifying the de-Poissonization procedure, based on the notion of JS-admissible functions (combining ideas from Jacquet and Szpankowski [12] and the classical paper by Hayman [7]). The following materials, different from those in [12], are modified from [6], where more details are provided.

Definition 1. *An entire function $\tilde{f}$ is said to be JS-admissible, denoted by $\tilde{f} \in \mathscr{JS}$, if the following two conditions hold for $|z| \geqslant 1$.*

(I) *There exist $\alpha, \beta \in \mathbb{R}$ such that uniformly for $|\arg(z)| \leqslant \varepsilon$,*

$$\tilde{f}(z) = O\left(|z|^{\alpha}(\log_+ |z|)^{\beta}\right),$$

where $\log_+ x := \log(1 + x)$.

(O) *Uniformly for* $\varepsilon \leqslant |\arg(z)| \leqslant \pi$,

$$f(z) := e^z \tilde{f}(z) = O\left(e^{(1-\varepsilon')|z|}\right).$$

Here and throughout this paper, the generic symbols $\varepsilon, \varepsilon'$ *denote small quantities whose values are immaterial and not necessarily the same at each occurrence.*

For convenience, we also write $\tilde{f} \in \mathscr{J}\mathscr{S}_{\alpha,\beta}$ to indicate the growth order of $\tilde{f}$ inside the sector $|\arg(z)| \leqslant \varepsilon$.

Note that if $\tilde{f}$ satisfies condition **(I)**, then, by Cauchy's integral representation for derivatives (or by Ritt's theorem; see [22, Ch. 1, § 4.3]), we have,

$$\tilde{f}^{(k)}(z) = O\left(|z|^{\alpha-k}(\log_+|z|)^\beta\right).$$

On the other hand, by Cauchy's integral representation, we also have

$$\begin{aligned} a_n &= \frac{n!}{2\pi i}\oint_{|z|=n} z^{-n-1}e^z\tilde{f}(z)\,\mathrm{d}z \\ &\approx \tilde{f}(n)\frac{n!}{2\pi i}\oint_{|z|=n} z^{-n-1}e^z\,\mathrm{d}z \\ &= \tilde{f}(n), \end{aligned}$$

since the saddle-point $z = n$ of the factor $z^{-n}e^z$ is unaltered by the comparatively more smooth function $\tilde{f}(z)$.

The latter analytic viewpoint provides an additional advantage of obtaining an expansion by using the Taylor expansion of $\tilde{f}$ at $z = n$, yielding

$$a_n = \sum_{j\geqslant 0}\frac{\tilde{f}^{(j)}(n)}{j!}\tau_j(n), \tag{25}$$

where

$$\tau_j(n) := n![z^n](z-n)^j e^z = \sum_{0\leqslant \ell\leqslant j}\binom{j}{\ell}(-1)^{j-\ell}\frac{n!n^{j-\ell}}{(n-\ell)!} \qquad (j = 0, 1, \dots),$$

and $[z^n]\phi(z)$ denotes the coefficient of z^n in the Taylor expansion of $\phi(z)$. We call such an expansion *the Poisson-Charlier expansion* since the τ_j's are essentially the Charlier polynomials $C_j(\lambda, n)$ defined by

$$C_j(\lambda, n) := \lambda^{-n}n![z^n](z-1)^j e^{\lambda z},$$

so that $\tau_j(n) = n^j C_j(n, n)$. For other terms used in the literature and more properties, see [8] and the references therein. In particular, the expansion (25) is absolutely convergent when $\tilde{f}$ is entire.

Proposition 2. *Assume* $\tilde{f} \in \mathscr{J}\mathscr{S}_{\alpha,\beta}$. *Let* $f(z) := e^z\tilde{f}(z)$. *Then the Poisson-Charlier expansion* (25) *of* $f^{(n)}(0)$ *is also an asymptotic expansion in the sense that*

$$\begin{aligned} a_n &:= f^{(n)}(0) = n![z^n]f(z) = n![z^n]e^z\tilde{f}(z) \\ &= \sum_{0\leqslant j<2k}\frac{\tilde{f}^{(j)}(n)}{j!}\tau_j(n) + O\left(n^{\alpha-k}(\log n)^\beta\right), \end{aligned}$$

for $k = 1, 2, \dots$.

The polynomial growth of condition **(I)** is sufficient for all our uses; see [12] for more general versions.

The real advantage of introducing admissibility is that it opens the possibility of developing closure properties as we now briefly discuss.

Lemma 4. *Let m be a nonnegative integer and $\alpha \in (0,1)$.*

(i) $z^m, e^{-\alpha z} \in \mathscr{J}\mathscr{S}$.

(ii) If $\tilde{f} \in \mathscr{J}\mathscr{S}$, *then* $\tilde{f}(\alpha z), z^m \tilde{f} \in \mathscr{J}\mathscr{S}$.

(iii) If $\tilde{f}, \tilde{g} \in \mathscr{J}\mathscr{S}$, *then* $\tilde{f} + \tilde{g} \in \mathscr{J}\mathscr{S}$.

(iv) If $\tilde{f} \in \mathscr{J}\mathscr{S}$, *then the product* $\tilde{P}\tilde{f} \in \mathscr{J}\mathscr{S}$, *where* $\tilde{P}$ *is a polynomial of* z.

(v) If $\tilde{f}, \tilde{g} \in \mathscr{J}\mathscr{S}$, *then* $\tilde{h} \in \mathscr{J}\mathscr{S}$, *where* $\tilde{h}(z) := \tilde{f}(\alpha z)\tilde{g}((1-\alpha)z)$.

(vi) If $\tilde{f} \in \mathscr{J}\mathscr{S}$, *then* $\tilde{f}' \in \mathscr{J}\mathscr{S}$, *and thus* $\tilde{f}^{(m)} \in \mathscr{J}\mathscr{S}$.

Proof. Straightforward and omitted. □

Specific to our need are the following transfer principles, first the real version and then the complex one.

Lemma 5. *Let $\tilde{f}(z)$ and $\tilde{g}(z)$ be entire functions satisfying*

$$\tilde{f}(z) = (1 - e^{-pz})\tilde{f}(pz) + e^{-pz}\tilde{f}(qz) + \tilde{g}(z), \tag{26}$$

with $\tilde{f}(0) = \tilde{g}(0) = 0$. *If* $\tilde{g}(x) = O(x^\alpha (\log_+ x)^\beta)$ *for real large x, where $\alpha, \beta \in \mathbb{R}$, then*

$$\tilde{f}(x) = \begin{cases} O(x^\alpha (\log_+ x)^\beta), & \text{if } \alpha > 0; \\ \left\{ \begin{array}{ll} O((\log_+ x)^{\beta+1}), & \text{if } \beta > -1 \\ O(\log_+ \log_+ x), & \text{if } \beta = -1 \\ O(1), & \text{if } \beta < -1 \end{array} \right\}, & \text{if } \alpha = 0; \\ O(1), & \text{if } \alpha < 0. \end{cases} \tag{27}$$

Proof. The idea of the proof here is that $\tilde{f}(x)$ behaves asymptotically like the following recurrence

$$\phi(x) = \phi(px) + \tilde{g}(x),$$

with $\phi(0) = 0$. To that purpose, we need only to show that $\tilde{f}(x)$ grows at most polynomially for large x. This is easily achieved by noticing that f is bounded above by the function defined by the *trie-recurrence*

$$\lambda(x) = \lambda(px) + \lambda(qx) + \nu(x),$$

with $\lambda(0) = \nu(0) = 0$, where

$$\nu(x) := \begin{cases} Kx^{\bar{\alpha}}, & \text{if } x > 1; \\ Kx, & \text{if } 0 \leqslant x \leqslant 1, \end{cases}$$

$K > 0$ being a large constant and $\bar{\alpha} := \max\{\lfloor\alpha\rfloor, 0\} + 1$. Note that the exact solution of λ is given by

$$\lambda(x) = \sum_{j,\ell \geqslant 0} \binom{j+\ell}{j} \nu\left(p^j q^\ell x\right).$$

We then deduce, from this, that $\lambda(x) = O(x^{\bar{\alpha}})$ for large x. Accordingly, $\tilde{f}$ is polynomially bounded. The more precise estimates (27) then follows from standard Mellin arguments (by subtracting the first few $\bar{\alpha}+1$ terms of the Taylor expansion of $\lambda(x)$ and then considering the Mellin transform of λ so truncated, which exits in the strip $-\bar{\alpha}-1 < \Re(s) < -\bar{\alpha}$). □

Proposition 3. *Let $\tilde{f}(z)$ and $\tilde{g}(z)$ be entire functions satisfying (26). Then*

$$\tilde{f} \in \mathscr{JS} \quad \text{if and only if} \quad \tilde{g} \in \mathscr{JS}.$$

Proof. The necessity part follows from Lemma 4. We prove the sufficiency, namely, if $\tilde{g} \in \mathscr{JS}$, then $\tilde{f} \in \mathscr{JS}$.

Write throughout the proof $z = re^{i\theta}$, $r \geqslant 0$ and $-\pi \leqslant \theta \leqslant \pi$. Consider first the region when $\varepsilon \leqslant |\theta| \leqslant \pi$. By assumption, $|e^z\tilde{g}(z)| \leqslant Ke^{(1-\varepsilon_1)r}$. Define

$$M(r) := \max_{\varepsilon \leqslant |\theta| \leqslant \pi} |f(z)| \qquad (r \geqslant 0).$$

Then by the functional equation

$$f(z) = \left(e^{qz} - e^{(q-p)z}\right) f(pz) + f(qz) + e^z\tilde{g}(z),$$

we have

$$M(r) \leqslant \left|e^{qz} - e^{(q-p)z}\right| M(pr) + M(qr) + Ke^{(1-\varepsilon_1)r}.$$

By using Pittel's inequality (see [23, Appendix])

$$|e^z - 1| \leqslant (e^r - 1)e^{-r(1-\cos\theta)/2} \qquad (r \geqslant 0; |\theta| \leqslant \pi),$$

we have

$$\begin{aligned}
\left|e^{qz} - e^{(q-p)z}\right| &= \left|e^{(q-p)z}\right| \left|e^{pz} - 1\right| \\
&\leqslant e^{(q-p)r\cos\theta}\left(e^{pr} - 1\right) e^{-pr(1-\cos\theta)/2} \\
&= \left(e^{qr} - e^{(q-p)r}\right) e^{-(q-p/2)r(1-\cos\theta)} \\
&\leqslant e^{-\varepsilon_2 r}\left(e^{qr} - e^{(q-p)r}\right),
\end{aligned} \tag{28}$$

for $\varepsilon \leqslant |\theta| \leqslant \pi$. Let $\varepsilon' := \min\{\varepsilon_1, \varepsilon_2\}$. It follows that

$$M(r) \leqslant e^{-\varepsilon' r}\left(e^{qr} - e^{(q-p)r}\right) M(pr) + M(qr) + Ke^{(1-\varepsilon')r}.$$

Let $\tilde{M}(r) := M(r)e^{-(1-\varepsilon')r}$. Then

$$\tilde{M}(r) \leqslant e^{-\varepsilon' pr}\left(1 - e^{-pr}\right)\tilde{M}(pr) + e^{-(1-\varepsilon')pr}\tilde{M}(qr) + K.$$

By the same bounding argument used in Lemma 5, we see that $\tilde{M}(r) = O(1)$, and thus $M(r) = O(e^{(1-\varepsilon')r})$. [Technically, we define a function, say $\phi(r)$, satisfying the functional equation

$$\phi(r) = e^{-\varepsilon' pr}\left(1 - e^{-pr}\right)\phi(pr) + e^{-(1-\varepsilon')r}\phi(qr) + K,$$

prove $\phi(r) = O(1)$ and then $M(r) \leqslant \phi(r)$.]

We now consider the sector $|\theta| \leqslant \varepsilon$. Since $\tilde{g}(z) = O\left(|z|^{\alpha}(\log_{+}|z|)^{\beta}\right)$ in this sector, we can then show that $\tilde{f}$ grows at most polynomially and is thus JS-admissible, details being omitted here. $\square$