Psi-series method for equality of random trees and quadratic convolution recurrences

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Abstract

An unusual and surprising expansion of the form

$$p_n = \rho^{-n-1}\left(6n + \tfrac{18}{5} + \tfrac{336}{3125}n^{-5} + \tfrac{1008}{3125}\,n^{-6} + \text{smaller order terms}\right),$$

as $n \to \infty$, is derived for the probability p_n that two randomly chosen binary search trees are identical (in shape, hence in labels of all corresponding nodes). A quantity arising in the analysis of phylogenetic trees is also proved to have a similar asymptotic expansion. Our method of proof is new in the literature of discrete probability and the analysis of

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algorithms, and it is based on the logarithmic psi-series expansions for nonlinear differential equations. Such an approach is very general and applicable to many other problems involving nonlinear differential equations; many examples are discussed in this article and several attractive phenomena are discovered.

Key words. Psi-series method, nonlinear differential equations, random trees, recursive structures, singularity analysis, asymptotic analysis.
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1 Introduction

The motivating problem. This paper was originally motivated by the following problem. Find the asymptotics of the sequence p_n defined recursively by

$$p_n = n^{-2} \sum_{0 \le j < n} p_j \, p_{n-1-j} \qquad (n \ge 1), \tag{1}$$

with the initial condition $p_0 = 1$. The sequence p_n is nothing but the probability that two randomly chosen binary search trees (BSTs) of size n are identical (having exactly the same shape and hence the same labels for corresponding nodes). It was first studied by Martínez in [33] as an auxiliary function for understanding the typical performance of the equality test of two random BSTs; see below for more background details. A minor variation of this sequence was encountered in the analysis of maximum agreement subtrees in [9] under the Yule-Harding model.

While shape parameters defined on a *single* random tree have been extensively studied in the literature for many varieties of trees, properties of statistics defined on *a pair* or *a d-tuple* of random trees received comparatively less attention, partly because of the intrinsic complexity of the underlying analytic problems. Yet many practical situations (such as tanglegrams) naturally lead to such a study, a typical example being the so-called “hereditary properties” or “recurrent properties”, which in turn cover the equality, root occurrence, simplification rules, reduction rules, “clashes” and others as special cases; see [2, 18, 33, 34, 38] for more details.

Recently, there has been also an increasing interest in statistics defined on two random combinatorial objects; see [7] and the references therein.

Random BSTs. For completeness, we describe now *binary search trees* (BSTs). Given a sequence of distinct items $[x_1, \dots, x_n]$ from some totally ordered domain, we can construct the corresponding BST as follows. If $n = 0$, then the tree is empty. If $n \ge 1$, then we place x_1 at the root; the remaining items are compared one after another with x_1, and recursively inserted into the left subtree of the root if they are smaller or into the right subtree if larger. The two subtrees are constructed recursively by the same procedure according to the original order of the items; see Figure 1 for an example.

By *random BSTs*, we assume that all $n!$ permutations of n distinct elements are equally likely, and construct the BST from a random permutation. Then we see that the root assumes the value j with probability $1/n$ for $j = 1, \dots, n$, which is also the probability that the size L_n of the left subtree of the root is $j - 1$.

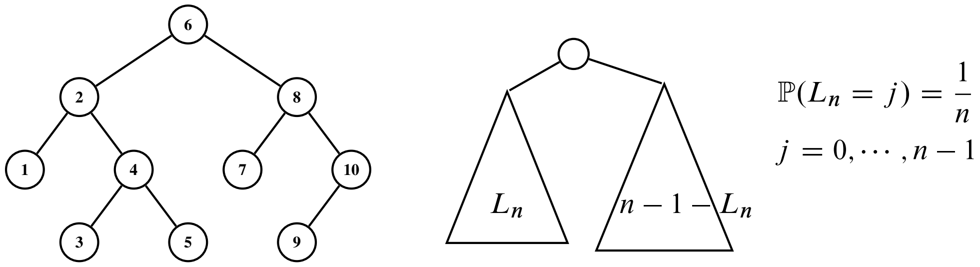
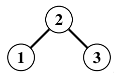


Figure 1: *Left: the BST constructed from the sequence* [6, 2, 4, 8, 7, 1, 5, 3, 10, 9]. *Right: the root assumes the value j with equal probability $1/n$ for $j = 1, \ldots, n$.*

Definition 1 (Equality of two ordered, labeled trees) *Two ordered, labeled trees of the same size (which is the total number of nodes) are said to be* equal *or* identical *if either both trees are empty or they have a common root label and all corresponding ordered subtrees are equal.*

The definition above can be easily generalized to the equality of d trees with $d \geq 2$.

Now we pick two random BSTs independently; then p_n equals the probability that the two trees are identical. Equivalently, we may pick two random permutations of n elements; then p_n denotes the probability that the BSTs constructed from these two permutations are equal. For example, the permutations (2, 1, 3) and (2, 3, 1) lead to the same BST



A simple upper bound. Take $n_0 \geq 1$ and

$$\rho := \min_{0 \leq j \leq n_0} \left(\frac{6n_0}{n_0 + 2} \cdot \frac{j+1}{p_j} \right)^{1/(j+1)};$$

we obtain by induction that

$$p_n \leq \frac{6n_0}{n_0 + 2}(n+1)\rho^{-n-1} \tag{2}$$

for all $n \geq 0$. This gives successively improving bounds for ρ for increasing values of n_0; see Table 1, where we take only the first four digits after the decimal point without rounding. In particular, taking $n_0 = 6$ leads to the bound $p_n \leq \frac{3}{2}(n+1)3^{-n}$. The simple bound (2) and

n_0	1	2	3	4	5	6	7	8	9
ρ	2	2.4494	2.6832	2.8284	2.9277	3	3.0274	3.0488	3.0659

n_0	10	20	30	40	50	60	70	80	90
ρ	3.0794	3.1235	3.1328	3.1362	3.1378	3.1387	3.1393	3.1396	3.1399

Table 1: *Numerical values of ρ.*

numerical evidence suggest the possibility that $p_n \sim 6n\rho^{-n-1}$ for some value $\rho \approx 3.14$ (see Figure 2), which is indeed the case as we will prove later.

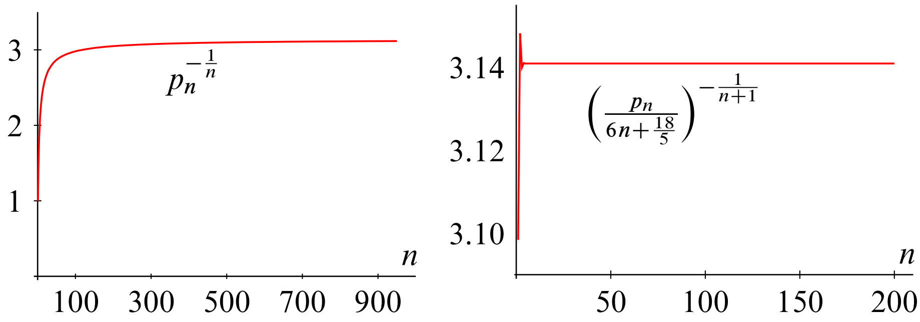


Figure 2: *The figures of* $p_n^{-1/n}$ *(left) and* $(p_n/(6n+\frac{18}{5}))^{-1/(n+1)}$ *(right).*

The nonlinear differential equation. Since the elementary argument that we have used above is not strong enough to derive more precise asymptotic approximations to p_n, we consider instead the generating function $P(z) := \sum_{n\geq 0} p_n z^n$, which satisfies the nonlinear *ordinary differential equation* (abbreviated throughout as ODE)

$$zP''(z) + P'(z) = P^2(z), \tag{3}$$

with the initial conditions $P(0) = P'(0) = 1$. This nonlinear ODE is of Emden-Fowler type for which there is no explicit closed form solution; see [36]. In addition to the apparent singularity determined by the equation, the ODE (3) also has *movable singularities* that are determined by the initial conditions. The reader is referred to Hille's book [28, Chap. 3] for a detailed discussion on singularities of ODEs.

Frobenius method. Starting from the ODE (3), the next step is often to try the Frobenius method (see [30]), namely, we assume the solution of $P(z)$ to be of the form

$$P(z) = \sum_{j\geq 0} c_j (1 - z/\rho)^{j-\alpha}, \tag{4}$$

for some α and $\rho > 0$, substitute this form into (3), and then determine α and the coefficients c_j inductively one after another. This classical procedure yields $\alpha = 2$, $c_0 = 6/\rho$,

$$c_1 = -\frac{12}{5\rho},\ c_2 = -\frac{7}{25\rho},\ c_3 = -\frac{14}{125\rho},\ c_4 = -\frac{63}{1250\rho},\ c_5 = -\frac{161}{9375\rho}. \tag{5}$$

But then an *inconsistency* arises since the coefficient of $(1 - z/\rho)^4$ of the left-hand side of (3) is

$$\rho^{-1}\left(12c_6 + \frac{483}{3125\rho}\right)$$

whereas the coefficient of $(1 - z/\rho)^4$ on the right-hand side is

$$\rho^{-1}\left(12c_6 + \frac{77}{625\rho}\right).$$

We see that the two quantities differ for whatever the value of c_6, which also means that c_6 cannot be determined by simply matching the coefficients in the usual way. This trial suggests that the local expansion of P near the singularity ρ will not be of the form (4) for some α and $\rho > 0$, and means that the classical Frobenius method fails for the nonlinear ODE (3).

Psi-series method. We consider now a different type of expansion called *logarithmic psi-series expansion* (or *Painlevé expansion*); see [28]. It will turn out that $P(z)$ admits an asymptotic expansion of the form

$$U(Z) := \sum_{j\ge 0} Z^{j-2} \sum_{0\le \ell\le \lfloor j/6\rfloor} c_{j,\ell}(\log Z)^{\ell}, \qquad Z := 1-\frac{z}{\rho}, \tag{6}$$

when z lies near the singularity ρ. Here and throughout this paper, $\log Z$ denotes its principal branch (which is real for $Z>0$). We will simply refer to such an expansion as a *psi-series* as no other types of psi-series will arise in this paper. The form (6), first conjectured by Martínez in [34, Ch. 9], is seen to be incompatible with (4). Indeed, $z=\rho$ is not a pole but instead *a pseudo-pole*; see [28]. The first few terms of $U(Z)$ are given as follows.

$$\begin{aligned}\rho U(Z) &= 6\,Z^{-2} - \frac{12}{5}\,Z^{-1} - \frac{7}{25} - \frac{14}{125}\,Z - \frac{63}{1250}\,Z^2 - \frac{161}{9375}\,Z^3 \\ &\quad + \rho c_6 Z^4 + \rho\sum_{j\ge 7}\sum_{0\le\ell\le\lfloor j/6\rfloor} c_{j,\ell} Z^{j-2}\log^{\ell} Z,\end{aligned}$$

for Z small, where $c_6 := c_{6,0}$ and the $c_{j,\ell}$'s are polynomials of the parameter $c_6\rho$ with degree $\lfloor (j-6\ell)/6\rfloor$ for $j\ge 7$. Note specially that $U(Z)$ is also a function of ρ and c_6.

Asymptotics of p_n. From the expansion (6) and suitable analytic continuation to be clarified later, we deduce our main result for p_n.

Theorem 1 *The probability p_n that two randomly chosen binary search trees of n nodes are equal satisfies the asymptotic expansion*

$$p_n \sim \rho^{-n-1}\left(6n + \frac{18}{5} + \sum_{j\ge 6} n^{-j+1} \sum_{0\le\ell<\lfloor j/6\rfloor} C_{j,\ell}(\log n)^{\ell}\right), \tag{7}$$

for explicitly computable constants $C_{j,\ell}$, where $\rho = 3.14085\,75672\,02936\,95160\ldots$

In particular, the first few terms read

$$\begin{aligned}p_n = \rho^{-n-1}\Bigg(&6n + \frac{18}{5} + \frac{336}{3125\,n^5} + \frac{1008}{3125\,n^6} + \frac{10416}{15625\,n^7} \\ &+ \frac{91728}{78125\,n^8} + \frac{8234352}{4296875\,n^9} + \frac{12228048}{4296875\,n^{10}} \\ &+ \frac{1}{n^{11}}\left(\frac{9483264}{5078125}\,H_n + \frac{5621191632}{726171875} + \frac{677376}{1625}c_6\right) + O\left(\frac{\log n}{n^{12}}\right)\Bigg),\end{aligned}$$

where $H_n := \sum_{1\le j\le n} j^{-1}$, and we see specially that *no terms of the form cn^{-j} with $j = 1,\ldots,4$ appear in the expansion.* Numerically, the parameter c_6 can be determined approximately as $c_6 = -0.00150\,84982\,09405\,93425\ldots$; see the numerical discussions on page 16 for details.

As far as we were aware, the asymptotic expansion (7) with missing terms is rare in the analysis of algorithms and applied probability literature. The expansion also indicates that the approximation of $p_n\rho^{n+1}$ by the first two terms $6n+\frac{18}{5}$ is numerically very precise as can be seen in Figure 2.

Features. In addition to the unusual form of (7) and its theoretical value *per se*, the interest of such a psi-series expansion is multifold. First, since no analytic form for the movable singularity ρ is available, the psi-series expansion provides an effective means of obtaining an approximate numerical value to ρ; see (25) on page 18 for more details. Second, from a methodological point of view, the method of proof we use to prove Theorem 1 is of some generality. Note that the first two terms on the right-hand side of (7) can be easily characterized by the method of matched coefficients once we assume that $P(z)$ has the form (6). Third, the precise approximation we derive has direct consequences in the original motivating problem, as well as several others in the examples we discuss below. Finally, while psi-series have long been used in many branches of mathematics and physics (see [28, 29]), little attention has been paid to the corresponding asymptotics of the coefficients, which themselves lead to unusual behaviors and unexpected phenomena (including asymptotic expansions with several missing terms), as we will see in the following sections.

A sketch of the method of proof. As indicated above, after checking the failure of the Frobenius method, we construct a suitable psi-series $U(Z)$ (by matching coefficients) so that U satisfies *formally* the ODE (3). Then our approach to proving Theorem 1 consists of the following steps (see Section 2 for details).

1. *Analytic continuation of P.* We first show that $P(z)$ can be analytically continued into the cut-region $\{z : |z| \le \rho+\varepsilon_0\}\setminus[\rho, \rho+\varepsilon_0]$ for some $\varepsilon_0 > 0$ (corresponding to the large circle in Figure 3), with the sole singularity $z = \rho$ there.

2. *Analytic nature of U.* The series in (6) defining $U(Z)$ is *a priori* an asymptotic expansion, but we will show that for any finite c_6, $U(Z)$ is absolutely convergent in some region containing particularly the region $\{Z : |Z||1-\log Z|^{1/6} \le \delta\}\setminus[-\delta_0, 0]$ (inside the dashed region in Figure 3) and defines an analytic function there.

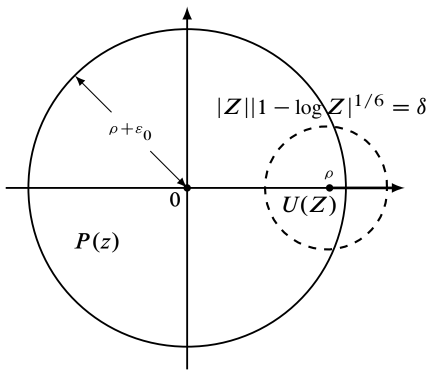


Figure 3: *Analyticity of P and U.*

3. *Equivalence of P and U.* The movable singularity ρ and the free parameter c_6 are uniquely determined once the initial conditions of the ODE (3) are given; the pair (ρ, c_6) in turn determines $U(Z)$, which is itself a function of z, ρ and c_6. The fact that P and U so determined have a common region of analyticity implies that P can be analytically continued through U, and, particularly, P has a psi-series expansion (6) near the dominant singularity ρ.

4. *Singularity analysis.* We then apply the singularity analysis (see [21]) and deduce (7).

This procedure is very general and we develop tools for dealing with more general situations. Note that no analytic forms for ρ and c_6 are available, so we will discuss numerical procedures to compute their values to high precision (see Section 2.4).

Outline of this paper. We describe the psi-series method and give the proof of the asymptotic expansion (7) in the next section. In Section 3 we consider several extensions of our analysis of the probability of equality of two random BSTs. More specifically, in Subsection 3.1 we obtain the probability of equality of $d > 2$ random BSTs. It turns out that the form of the asymptotic expansion for the probability of equality of d random BSTs differs drastically according to the parity of d, a result not intuitively obvious at all. Subsection 3.2 considers the case of two random m-ary search trees and we will see that the number of missing terms in the asymptotic expansion increases as m grows. Equality of two random fringe-balanced BSTs is considered in Subsection 3.3 and there, unlike m-ary search trees, the error term beyond the constant term in the asymptotic expansion does not change with the structural parameter once it exceeds one, another unexpected result. Asymptotics of some quadratic recurrences of Faltung (convolution) type will then be considered in Section 4 with a few representative examples taken from the cost of partial-match queries in random trees, random partition structures and solutions of Boltzmann equations (from statistical physics). Section 5 contains our conclusions. A table (Table 3) summarizing all DEs studied in this paper is also given there. Finally, we group the technical details of some proofs in the Appendices at the end of the paper.

Notations. For each problem studied, ρ always denotes the dominant singularity of the associated nonlinear ODE and $Z := 1 - z/\rho$. The symbols $c, c', c_j, c'_j, c_{i,j}, C, C_j, C'_j, C_{i,j}, K, K'$ all denote suitably chosen constants, not necessarily the same at each occurrence. Similarly, $\varepsilon, \varepsilon', \varepsilon_j$ represent arbitrarily small positive constants.

2 The psi-series method

We discuss in detail the psi-series solution to our nonlinear ODE (3) and the tools needed to justify it, then we prove Theorem 1, following the procedure sketched in the Introduction.

2.1 Analytic properties of $P(z)$ and the ARS algorithm

First, the solution $P(z)$ to the ODE (3) has positive radius of convergence and is analytic at the apparent fixed singularity $z = 0$ by definition. By induction as we discussed in the introduction (Section 1) and Pringsheim's theorem (since all coefficients p_n are positive; see [23, p. 240] or [28, §1.8]), we expect that $P(z)$ has a finite movable singularity at, say $z = \rho$, and the asymptotics of p_n will be dictated by the local asymptotic expansion of $P(z)$ as $z \sim \rho$.

Martínez [34, p. 117] proved that the function $P(z)$, originally defined only inside the disk $|z| < \rho$ can be analytically continued to the cut-disk $\{z : |z| \le \rho + \varepsilon, z \notin [\rho, \rho + \varepsilon]\}$.

From a theoretical point of view, the movable singularity ρ for the ODE (3) can be either of the following types: (*i*) poles, (*ii*) branch points (algebraic or logarithmic), (*iii*) essential singularity. Simple poles and algebraic points are first excluded because of our previous trial via the Frobenius method. We then show that P can be analytically continued into a function defined by a series expansion of the form (6) that converges absolutely in some region covering the cut-region $\mathscr{C}_\varepsilon$ defined by

$$\mathscr{C}_\varepsilon := \{z \,:\, 0 < |z-\rho| \le \varepsilon, z \notin [\rho, \rho+\varepsilon]\}, \tag{8}$$

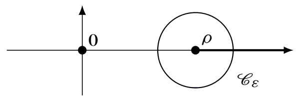


Figure 4: *The region* $\mathscr{C}_\varepsilon$.

Thus the possibility that ρ is an essential singularity is further excluded, and ρ is a logarithmic branch point (or called pseudo-pole).

In this paper, we first focus on the determination of the right form of the solution to (3). More detailed and complete introduction and discussions on the theory related to *Painlevé analysis* can be found in [11, 13] and the references therein.

The ARS method (Type checking). A widely used procedure to check the singularity type (and the local expansion) of nonlinear ODEs is the following procedure, often called the ARS algorithm due to Ablowitz, Ramani and Segur [1], which bears some resemblance to the Frobenius method.

This method starts from assuming that the solution to the ODE (3) admits the formal Laurent expansion (4) about the cut-disk $\mathscr{C}_\varepsilon$ for some positive number ε.

❶ *Leading order analysis*: Assume $P(z) \sim c_0(1-z/\rho)^{-\alpha}$ for z near the dominant singularity ρ. By balancing the dominant terms $\rho P''(z)$ and $P(z)^2$ in (3), we see, as in the Frobenius method, that $\alpha = 2$ and the companion constant $c_0 = 6/\rho$. Thus we can exclude the possibility of a dominant algebraic singularity.

❷ *Resonance analysis*: Starting from the pair $(\alpha, c_0) = (2, 6/\rho)$, if the solution admits only poles, then by substituting (4) into (3) and by equating coefficients, we see that the coefficients c_j's are characterized by the recurrence relation of the form

$$\Phi(j)c_j = (j-3)^2 c_{j-1} + \rho \sum_{1\le \ell < j} c_\ell c_{j-\ell} =: \Psi_j(\rho, c_0, c_1, \ldots, c_{j-1}), \tag{9}$$

for $j \ge 1$, where $\Phi(j) = (j+1)(j-6)$ and $c_j = 0$ for all $j < 0$. The roots of $\Phi(j)$ are called *resonance* and -1 is always a root of $\Phi(j)$, reflecting the *arbitrariness of the movable singularity* ρ. Alternatively, a less involved and very commonly used technique is to substitute the test function

$$c_0(1-z/\rho)^{-\alpha} + c_r(1-z/\rho)^{r-\alpha} \tag{10}$$

into the ODE (3) instead. By collecting the coefficients corresponding to the term $c_r(1-z/\rho)^{r-4}$, we still get the same α, c_0 and $\Phi(r)$. In this case, we see that Φ has only one positive resonance 6 that needs to be further examined.

❸ *Compatibility*: Once we have the system (9) and identify the resonance, the next step is to consider its solvability. Obviously, (4) is the solution to (3) if and only if all the coefficients c_k's can be computed recursively by (9). This fact defines the *compatibility of the resonance*: if $\Psi_r(\rho, c_0, c_1, \ldots, c_{r-1}) = 0$ for any resonance r of Φ, then the resonance r is said to be compatible; otherwise, r is incompatible.

From the recurrence (9) it is straightforward to prove that $r = 6$ is incompatible. The formal series solution by introducing suitable logarithmic terms starting at the index 6 has to be considered instead (see (6)). The movable singularity ρ to (3) is proved to be a logarithmic branch point since we will show that the associated series solution is absolutely convergent in some cut-region.

In cases when all resonances are compatible, the Laurent expansion is the solution we need by the Frobenius method. The above ARS Algorithm is useful in determining if a nonlinear ODE admits the *Painlevé property*, namely, the only movable singularities of the ODE are poles. In our case, the ODE (3) does not satisfy the Painlevé property.

Our approach vs the ARS algorithm. The method of proof we use does not, however, rely completely on this method for two reasons. First, it requires the *a priori* information that ρ is not an essential singularity, a property often hard to justify. Second, even if we can prove that the singularity is not essential, the incompatibility of a resonance (or several) may in some cases be very difficult to establish due to the variation of an additional parameter as in the cases of d random BSTs (Subsection 3.1) and m-ary search trees (Subsection 3.2).

On the other hand, the ARS algorithm does provide an effective means of computing the exact form of the psi-series expansion for all the examples we discuss, notably the characterization of the resonance. We will thus use the ARS algorithm for two purposes: first, when the resonance equation has no positive integral resonance or when all resonances are compatible, then the solution is given by a Laurent expansion; second, when Laurent expansion fails, we use the ARS algorithm to guess the possible form of the psi-series expansion we are looking for, and then the proof will follow the same line we use for p_n. Of course, there are also cases for which the ARS algorithm can be easily justified and the singularity is not essential (say, by the absolute convergence of the psi-series).

2.2 Analytic continuation of $P(z)$

To prove that any solution to the ODE (3) can be analytically continued outside $|z| < \rho$, we show that any non-real singularity of (3) has a modulus larger than ρ. Note that (3) can be written as

$$\big(zP'(z)\big)' = P(z)^2. \tag{11}$$

Proposition 1 ([34] Lemma 9.1) *The solution $P(z)$ of the ODE (3) can be analytically continued to a cut-disk $\{z : |z| \le \rho + \varepsilon_0\} \setminus [\rho, \rho + \varepsilon_0]$ for some $\varepsilon_0 > 0$.*

Proof. We show that if $\rho_\theta e^{\theta \mathrm{i}}$ is a singularity of (11), where $\rho_\theta > 0$ and $\theta \neq 0$, then $\rho_\theta > \rho$. Although the Proposition was proved in [34], its proof is given in detail since the same argument will be applied and extended later.

We start from defining three functions : $v(x) = xP'(x)$, $\varpi(x) = \left|P\left(xe^{\theta \mathrm{i}}\right)\right|$, and $w(x) = \left|zP'(z)\right|_{z=xe^{\theta \mathrm{i}}}$, $x \ge 0$, with $\varpi(0) = P(0), w(0) = v(0)$. Then these three functions are continuous, increasing and satisfy the following relations

$$\begin{aligned}
&v'(x) = P(x)^2,\\
&\varpi(x) < P(x), \quad |\varpi'(x)| \le \frac{w(x)}{x} \le \frac{v(x)}{x},\\
&w(x) < v(x), \quad |w'(x)| \le \varpi(x)^2 \le v'(x),
\end{aligned}$$

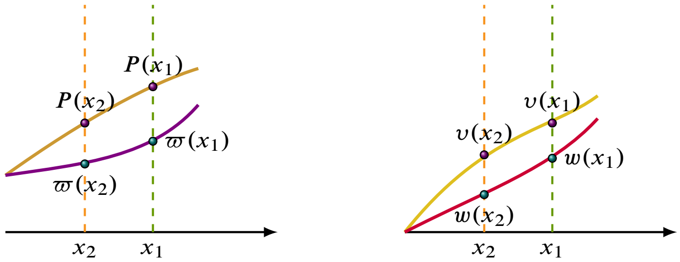


Figure 5: *A continuity argument used to compare the orders of the two functions* $P(x)$ *and* $\varpi(x)$ *(left), and those of the two functions* $\upsilon(x)$ *and* $w(x)$*.*

for $x > 0$. Note that both derivatives ϖ' and w' exist.

By continuity, we choose $0 < x_2 < x_1 < \rho$ such that $w(x_1) < \upsilon(x_2) < \upsilon(x_1)$ and $\varpi(x_1) < P(x_2) < P(x_1)$ (see Figure 4). We show first that

$$P(x_2 + t) > \varpi(x_1 + t) \quad \text{for } t \in J := [0, \min\{\rho_\theta - x_1, \rho - x_2\}]. \tag{12}$$

It follows from this inequality that $\rho_\theta - x_1 \geq \rho - x_2$, for, otherwise,

$$\infty > P(x_2 + \rho_\theta - x_1) > \varpi(x_1 + \rho_\theta - x_1) = \infty,$$

a contradiction. This implies that that $\rho_\theta \geq \rho + (x_1 - x_2) > \rho$.

We prove (12) by *reductio ad absurdum*. Assume that (12) is false. Then we can find the smallest positive number t_1 such that

$$P(x_2 + t_1) = \varpi(x_1 + t_1), \quad P(x_2 + t) > \varpi(x_1 + t) \quad t \in [0, t_1).$$

Then $P'(x_2 + t_1) \leq \varpi'(x_1 + t_1)$ implies that

$$\upsilon(x_2 + t_1) = (x_2 + t_1)P'(x_2 + t_1) < (x_1 + t_1)\varpi'(x_1 + t_1) \leq w(x_1 + t_1).$$

This, together with the relation $\upsilon(x_2) > w(x_1)$, implies that we can find the minimum $t_0 \in (0, t_1]$ such that

$$\begin{aligned} \upsilon(x_2 + t_0) = w(x_1 + t_0) &= w(x_1) + \int_0^{t_0} w'(x_1 + s)\,\mathrm{d}s \\ &< \upsilon(x_2) + \int_0^{t_0} \varpi(x_1 + s)^2\,\mathrm{d}s \\ &< \upsilon(x_2) + \int_0^{t_0} P(x_2 + s)^2\,\mathrm{d}s \\ &= \upsilon(x_2) + \int_0^{t_0} \upsilon'(x_2 + s)\,\mathrm{d}s = \upsilon(x_2 + t_0), \end{aligned}$$

which is absurd. We thus proved (12) and the Proposition. ▮

2.3 Analyticity of the psi-series

We next prove that $U(Z)$, as defined in (6), is analytic in a region covering particularly the cut-disk $\mathscr{C}_\varepsilon$ (defined in (8)) for some positive $\varepsilon > 0$. Recall that $U(Z)$ is also a function of ρ (the dominant movable singularity of $P(z)$) and c_6.

Proposition 2 *For any pair of numbers $c_6, \rho \in \mathbb{C}$, there is a constant $B > 0$ so that the psi-series $U(Z)$ as given in (6) defines an analytic function in the region*

$$\mathscr{Z} := \{Z : -\pi < |\arg Z| \le \pi, B|Z||1 - \log Z|^{1/6} < 1\} \setminus [-x_0, 0],$$

where x_0 solves the equation $Bx|1 - \log x + \pi \mathrm{i}|^{1/6} = 1$ with real x.

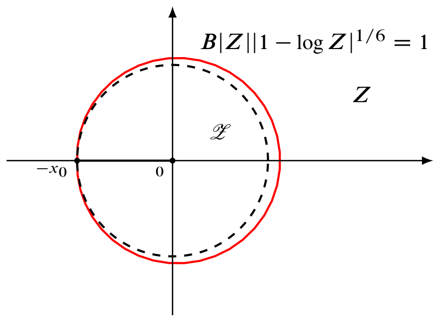


Figure 6: *$\mathscr{Z}$ is the region inside the red curve $B|Z||1 - \log Z|^{1/6} = 1$ (the dashed-line represents the circle with radius x_0).*

We will prove that the psi-series (6) converges absolutely for $Z \in \mathscr{Z}$. Then the analyticity of U follows from term by term differentiation. To that purpose, we modify an approach due to Hille [26] with some new ingredients; see also [15, 25, 27]. This approach is summarized as follows.

Re-summation. The psi-series (6) is in essence a double series

$$U(Z) = \sum_{j,\ell \ge 0} c'_{j,\ell} Z^{j+6\ell-2} (\log Z)^{\ell},$$

and we will group them first in increasing powers of Z, leading to polynomials of $\log Z$ as coefficients.

Linear system. Since first-order ODEs are easier to solve in general, we rewrite the ODE (3) as a first-order system; see (13). Then the coefficients (polynomials in $\log Z$) appearing in the previous re-summation step satisfy a first-order nonhomogeneous linear system, which can then be solved in a form suitable for majorization purposes.

A uniform bound. We then prove by induction a simple uniform bound for the coefficients (see (19)). The absolute convergence then follows.

This method of proof can be readily extended to cover all types of ODEs we discuss in this paper, whatever their orders are.

From Propositions 1 and 2, we see that if we equate $P(z_0) = U(Z_0)$, where $Z_0 = 1 - z_0/\rho$, for z_0 in their common region of analyticity, then the solution $P(z)$ can be analytically continued to at least the region (see Figure 3 on Page 6)

$$\{z : |z| \le \rho + \varepsilon_0\} \cup \left\{ z : B\left|1 - \frac{z}{\rho}\right| \left|1 - \log\left(1 - \frac{z}{\rho}\right)\right|^{1/6} < 1 \right\} \setminus [\rho, (1 + x_0)\rho].$$

We then deduce (7) by the singularity analysis of Flajolet and Odlyzko [21].

Recurrence of u_k. We first rewrite the ODE (3) for P into that for U, which becomes

$$\big((1-Z)U'(Z)\big)' = \rho U(Z)^2.$$

For convenience, let $U_0 = \rho U$. Then

$$\big((1-Z)U_0'(Z)\big)' = U_0(Z)^2.$$

As in [26], we then convert this ODE into a first-order differential system by introducing an additional function $V_0 := (1-Z)U_0'(Z)$ as follows.

$$\begin{cases} U_0'(Z) = \dfrac{V_0(Z)}{1-Z}, \\ V_0'(Z) = U_0(Z)^2. \end{cases} \tag{13}$$

Write $\tau = \log Z$, where $\log Z$ denotes its principal branch. Let $U_0(Z) = \sum_{k\geq 0} u_k(\tau)Z^{k-2}$ and $V_0(Z) = \sum_{k\geq 0} v_k(\tau)Z^{k-3}$, where u_k and v_k are polynomials in τ of degree at most $\lfloor k/6 \rfloor$. Note that $(\mathrm{d}\tau)/(\mathrm{d}Z) = Z^{-1}$ and $u_0(\tau) = 6$. From (13), we derive an infinite system of equations in k ($\dot{u}_k := u_k'(\tau)$)

$$\begin{cases} \dot{u}_k + (k-2)u_k = v_k + \displaystyle\sum_{0\leq j<k} v_j, \\ \dot{v}_k + (k-3)v_k = 12u_k + \displaystyle\sum_{1\leq j<k} u_j u_{k-j}, \end{cases} \qquad (k \geq 7)$$

with the initial values ($v_k = \dot{u}_k + (k-2)u_k - \dot{u}_{k-1} - (k-3)u_{k-1}$)

u_0	u_1	u_2	u_3	u_4	u_5	u_6
6	$-\frac{12}{5}$	$-\frac{7}{25}$	$-\frac{14}{125}$	$-\frac{63}{1250}$	$-\frac{161}{9375}$	$c_6\rho - \frac{14\tau}{3125}$
v_0	v_1	v_2	v_3	v_4	v_5	v_6
-12	$\frac{72}{5}$	$-\frac{12}{5}$	$-\frac{14}{125}$	$\frac{7}{625}$	$\frac{154}{3125}$	$\frac{147}{3125} + 4c_6\rho - \frac{56\tau}{3125}$

(14)

We can further express the above system in terms of matrices as follows. Let

$$\boldsymbol{\phi}_k := \begin{pmatrix} u_k \\ v_k \end{pmatrix}, \ \mathbf{A}_k := \begin{pmatrix} k-2 & -1 \\ -12 & k-3 \end{pmatrix}, \text{ and } \mathbf{g}_k := \begin{pmatrix} \displaystyle\sum_{0\leq j<k} v_j \\ \displaystyle\sum_{1\leq j<k} u_j u_{k-j} \end{pmatrix}.$$

Then, we obtain the nonhomogeneous linear differential system

$$\dot{\boldsymbol{\phi}}_k + \mathbf{A}_k\boldsymbol{\phi}_k = \mathbf{g}_k, \qquad (k \geq 7). \tag{15}$$

We now derive a better integral form for our estimation purposes. To state the representation, we introduce the following norm: for any $\mathbf{x} \in \mathbb{C}^n$ and any matrix $\big(a_{ij}\big)_{n\times n}$,

$$\|\mathbf{x}\| = \max_{1\leq j\leq n} \{|x_j|\}, \quad \| \big(a_{ij}\big)_{n\times n} \| = \max_{1\leq j\leq n} \left\{ \sum_i |a_{ij}| \right\}, \tag{16}$$

which is the operator norm corresponding to the chosen norm on $\mathbb{C}^n$.

Lemma 1 *For $k \geq 7$, the nonhomogeneous linear differential system (15) admits a unique solution satisfying*

$$\lim_{\tau \to -\infty} \|e^{\mathbf{A}_k \tau} \boldsymbol{\phi}_k(\tau)\| = 0$$

of the form

$$\boldsymbol{\phi}_k(\tau) = \int_0^\infty \mathbf{P}\, e^{-x\mathbf{D}} \mathbf{P}^{-1} \mathbf{g}_k(\tau - x)\, \mathrm{d}x, \tag{17}$$

where $\mathbf{D} := \begin{pmatrix} k+1 & 0 \\ 0 & k-6 \end{pmatrix}$, $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ -3 & 4 \end{pmatrix}$ *and* $\mathbf{P}^{-1} = \begin{pmatrix} \frac{4}{7} & -\frac{1}{7} \\ \frac{3}{7} & \frac{1}{7} \end{pmatrix}$.

Proof. The fundamental matrix solution associated with the homogeneous part of (15) is $e^{-\tau \mathbf{A}_k}$, so we can solve (15) by multiplying it by $e^{x\mathbf{A}_k}$, using the fact that $u_k(\tau)$ and $v_k(\tau)$ are polynomials in τ, and then by integrating both sides from $-\infty$ to τ, yielding

$$\boldsymbol{\phi}_k(\tau) = \int_{-\infty}^{\tau} e^{(x-\tau)\mathbf{A}_k} \mathbf{g}_k(x)\, \mathrm{d}x = \int_0^\infty e^{-x\mathbf{A}_k} \mathbf{g}_k(\tau - x)\, \mathrm{d}x.$$

This proves the lemma. ∎

A uniform estimate for $\|\boldsymbol{\phi}_k\|$. With the operator norm defined in (16), we then have the estimates

$$\begin{aligned}
\max\{|u_k(\tau)|, |v_k(\tau)|\} &= \|\boldsymbol{\phi}_k(\tau)\| \\
&= \left\| \int_0^\infty \mathbf{P} e^{-x\mathbf{D}} \mathbf{P}^{-1} \mathbf{g}_k(\tau - x) \mathrm{d}x \right\| \\
&\leq \|\mathbf{P}\| \|\mathbf{P}^{-1}\| \int_0^\infty e^{-x(k-6)} \|\mathbf{g}_k(\tau - x)\| \mathrm{d}x \\
&\leq 5 \int_0^\infty e^{-x(k-6)} \max \Bigg\{ \sum_{0 \leq j < k} \left| v_j(\tau - x) \right|, \\
&\qquad\qquad \sum_{1 \leq j < k} \left| u_j(\tau - x) u_{k-j}(\tau - x) \right| \Bigg\} \mathrm{d}x.
\end{aligned} \tag{18}$$

For a fixed $B > 0$, define the region $\mathscr{T} = \mathscr{T}_B$ in the τ-plane

$$\mathscr{T} := \left\{ \tau \;:\; B|1 - \tau|^{1/6} e^{\Re(\tau)} < 1, \right.$$
$$\left. \Re(\tau) < 0, -\pi < \theta \leq \pi \right\},$$

which corresponds to the region inside the blue curve in the left figure of Figure 7, the corresponding region in the z-plane being shown in the right figure (see also Figure 6).

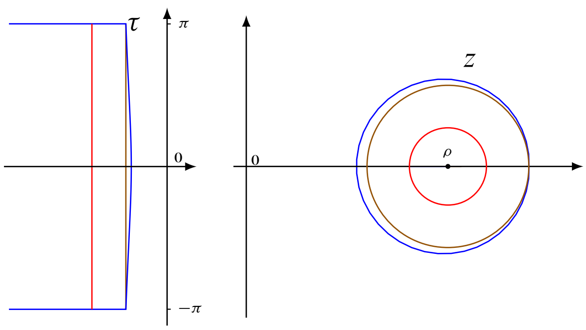


Figure 7: *Region $\mathscr{T}$ in both τ- and z-planes.*

Now write $\tau = \xi + \theta\mathrm{i}$, where $-\pi < \theta \le \pi$, so that $\tau \in \mathscr{T}$ implies that $B|1-\tau| > B$. We prove by induction that

$$\|\boldsymbol{\phi}_k(\tau)\| \le \frac{C_0}{(k+1)^2} B^k |1-\tau|^{k/6}, \tag{19}$$

for $k \ge 0$ and $\tau \in \mathscr{T}$, where $C_0 \ge \max\{|u_0|, |v_0|\} = 12$ and the constant B is, for convenience, assumed to be ≥ 2 (and to be specified later). The exact exponent of the polynomial factor $1/(k+1)^2$ here is less important and chosen simply for numerical convenience; it can be replaced by other powers of $k+1$.

First, by induction hypothesis and $B|1-\tau|^{1/6} \ge 2$,

$$\begin{aligned}
\sum_{0\le j<k} \left|v_j(\tau)\right| &\le C_0 \sum_{0\le j<k} \frac{B^j\,|1-\tau|^{j/6}}{(j+1)^2} \\
&\le C_0 B^k |1-\tau|^{k/6} \sum_{1\le j\le k} \frac{(B|1-\tau|^{1/6})^{-j}}{(k-j+1)^2} \\
&\le C_0 B^k |1-\tau|^{k/6} \sum_{1\le j\le k} \frac{2^{-j}}{(k-j+1)^2} \\
&\le \frac{C_0 C_1(k)}{(k+1)^2} B^k |1-\tau|^{k/6},
\end{aligned}$$

where $C_1(k) := \sum_{1\le j\le k} \frac{2^{-j}(k+1)^2}{(k-j+1)^2}$ is a bounded sequence for all $k \ge 0$, and

$$\begin{aligned}
\sum_{1\le j<k} \left|u_j(\tau)u_{k-j}(\tau)\right| &\le C_0^2 B^k |1-\tau|^{k/6} \sum_{1\le j<k} \frac{1}{(j+1)^2(k-j+1)^2} \\
&\le \frac{C_0^2 C_2(k)}{(k+1)^2} B^k |1-\tau|^{k/6},
\end{aligned}$$

where $C_2(k) := \sum_{1\le j<k} \frac{(k+1)^2}{(j+1)^2(k-j+1)^2}$ is also bounded for all k. Note that for large k

$$\begin{cases} C_1(k) \sim 1, \\ C_2(k) \sim \frac{\pi^2}{3} - 2 \approx 1.29. \end{cases} \tag{20}$$

Also by a direct partial fraction expansion, we deduce that

$$\begin{aligned}
C_2(k) &\le \frac{2(k+1)^2}{(k+2)^2} \sum_{2\le j\le k} \frac{1}{j^2} + \frac{4(k+1)^2}{(k+2)^3}(H_k - 1) \\
&\le \frac{\pi^2}{3} - 2 + \frac{4}{k+2}(H_k - 1).
\end{aligned}$$

Thus, by (18),

$$\begin{aligned}
\|\boldsymbol{\phi}_k(\tau)\| &\le \frac{5C_0\left(C_0C_2(k) \vee C_1(k)\right)}{(k+1)^2} B^k \int_0^\infty e^{-x(k-6)} |1-\tau+x|^{k/6} \mathrm{d}x \\
&\le \frac{5C_0\left(C_0C_2(k) \vee C_1(k)\right)}{(k-6)(k+1)^2} B^k |1-\tau|^{k/6} \int_0^\infty e^{-x} \left|1 + \frac{x}{(k-6)(1-\tau)}\right|^{k/6} \mathrm{d}x,
\end{aligned}$$

where $x \vee y := \max\{x, y\}$. Since $\Re(\tau) < 0$ whenever $\tau \in \mathscr{T}$, we see that

$$\begin{aligned}\int_0^\infty e^{-x} \left|1 + \frac{x}{k(1-\tau)}\right|^{(k+6)/6} \mathrm{d}x &\le \int_0^\infty e^{-x} \left(1 + \frac{x}{k}\right)^{(k+6)/6} \mathrm{d}x \\ &\le \int_0^\infty e^{-\frac{5}{6}x} \left(1 + \frac{x}{k}\right) \mathrm{d}x \\ &= \frac{6}{5} + \frac{36}{25k}.\end{aligned}$$

On the other hand, since $2^j \ge (j+1)^2$ for $j \ge 6$, we see that

$$\begin{aligned}C_1(k) &\le \sum_{1\le j\le 5} \left(\frac{1}{2^j} - \frac{1}{(j+1)^2}\right)\left(\frac{k+1}{k-j+1}\right)^2 + C_2(k) + \frac{(k+1)^2}{2^k} \\ &\le 5C_2(k) \le C_0 C_2(k),\end{aligned}$$

for all $k \ge 7$. Thus, it follows that

$$\begin{aligned}\|\boldsymbol{\phi}_k(\tau)\| &\le \frac{6C_0^2 C_2(k)}{(k-6)(k+1)^2}\left(1 + \frac{6}{5(k-6)}\right) B^k |1-\tau|^{k/6} \\ &\le \frac{C_0}{(k+1)^2} B^k |1-\tau|^{k/6},\end{aligned}$$

for $k \ge k_0$, where k_0 is chosen to be the least positive integer ≥ 7 determined (independently of B) by the equation

$$6C_0 C_2(k)\left(1 + \frac{6}{5(k-6)}\right) \le k - 6,$$

for $k \ge k_0$. Such a k_0 does exist since $C_2(k)$ is bounded for all $k \ge 0$. Indeed, for all $k \ge 0$

$$C_2(k) \le \tfrac{45321361290953861}{29873547265233672} \approx 1.517.$$

So for $m \ge k_0$, we have proved that "(19) holds for all $k < m$" implies that "(19) holds for $k = m$." The induction proof is complete if we choose a suitable B such that (19) holds for all $0 \le k < k_0$.

For more precise numerical purposes, we can use the following arguments. First, we have the inequality

$$\sum_{1\le j<k} \frac{(k+1)^2}{(j+1)^2(k-j+1)^2} = C_2(k) \le \frac{k-6}{6C_0\left(1 + \frac{6}{5(k-6)}\right)},$$

and we choose the least positive k such that the inequality is satisfied (see (20)). Numerically, the exact location, based on this estimate, is very easy to identify. It suffices to find the least positive integer k_0 such that

$$\sum_{1\le j<k_0} \frac{(k_0+1)^2}{(j+1)^2(k_0-j+1)^2} - \frac{k_0-6}{6C_0\left(1 + \frac{6}{5(k_0-6)}\right)} < 0.$$

Once k_0 is determined, we can compute the first k_0 terms of u_k and v_k directly from (15) or (17). Note that u_k and v_k are polynomials in τ of degree at most $\lfloor k/6 \rfloor$, and any polynomial $u_k(\tau)$ can be expressed as $u_k(\tau) = \sum_{0 \le i \le \lfloor k/6 \rfloor} \hat{u}_{k,i}(1-\tau)^i$. Then a very crude upper bound of $|u_k|$ is given by

$$|u_k(\tau)| \le \left(\frac{k}{6}+1\right)\left(\max_{0\le i\le \lfloor \frac{k}{6}\rfloor} |\widehat{u}_{k,i}|\right)|1-\tau|^{k/6}.$$

since $\Re(\tau) < 0$. A similar bound also holds for $|v_k|$. Thus if we define

$$\begin{aligned} B := &\max_{1\le k\le 5}\left(\frac{(k+1)^2}{C_0}(|u_k| \vee |v_k|)\right)^{1/k} \\ &\vee \max_{6\le k\le k_0}\left(\frac{(k+1)^2}{C_0}\left(\frac{k}{6}+1\right)\max_{0\le i\le \lfloor k/6\rfloor}\left\{|\widehat{u}_{k,i}| \vee |\widehat{v}_{k,i}|\right\}\right)^{1/k}. \end{aligned}$$

Then (19) holds for all $0 \le k \le k_0$. Note that such bounds for B are overestimates but sufficient for our uses. This proves (19) for $k \ge 0$.

Numerically, we get the following table when the initial conditions of u_j and v_j are given in (14) with $\rho \approx 3.140857$ and $c_6 \approx -0.001508$.

C_0	12	18	24	28	28.8	36.3
k_0	109	156	204	235	242	300
B	4.8	3.2	2.4	2.05	2	1.59

A similar argument applies when $1 < B < 2$. We then see that $B \to 1$ when C_0 increases (at the price of increasing k_0). Indeed, the value of B is determined by the first few terms of $|u_k|, |v_k|$ because $x^{1/k} \to 1$ as $k \to \infty$ for $x > 0$.

Absolute convergence of the psi-series. From (19), we obtain

$$\begin{aligned} \rho|U(Z)| &= \left|\sum_{k\ge 0} u_k(\tau)e^{(k-2)\tau}\right| \\ &\le C_0\, e^{-2\Re(\tau)} \sum_{k\ge 0} \frac{B^k |1-\tau|^{k/6} e^{k\Re(\tau)}}{(k+1)^2}. \end{aligned}$$

By d'Alembert's ratio test, if

$$B|1-\tau|^{1/6}e^{\Re(\tau)} < 1,$$

which is implied by $\tau \in \mathscr{T}$, then the series $\sum_{k\ge 0} u_k(\tau)e^{k\tau}$ is absolutely convergent. This completes the proof of Proposition 2. ∎

2.4 Numerical approximations to ρ and c_6

As mentioned in the Introduction, the function $U(Z)$ is also a function of ρ and c_6, which are themselves determined by the ODE (3) satisfied by P and the initial conditions. Thus equating

$P(z_0) = U(Z_0)$, where $Z_0 := 1 - z_0/\rho$, for z_0 lying in their common region of analyticity (see Figure 3) provides an analytic continuation of P (through U).

However, this description does not provide directly an effective numerical procedure for computing the values of the pair (ρ, c_6). We thus convert the numerical problem into an initial-value problem as follows. To fix U in a unique way, we connect $P(z)$ and $U(Z)$ by first choosing a number $z_0 \in [\varepsilon\rho, \rho-\varepsilon]$, and by considering the solution (ρ, c_6) of the two equations

$$\begin{cases} U(Z_0) = P(z_0), \\ U'(Z_0) = -\rho P'(z_0). \end{cases} \tag{21}$$

By Proposition 2 and simple upper and lower bounds for ρ, we see that, as a standard initial-value problem, the system of equations (21) has a unique solution pair for (ρ, c_6). This determines uniquely the pair (ρ, c_6), and accordingly $U(Z)$, which then provides the asymptotic solution we have been looking for.

For numerical purposes, we can compute the approximate values of $P(z_0)$ or $P'(z_0)$ by their corresponding truncated series expansions using, say the first N terms; for example, $P(z_0) \approx \sum_{j<N} p_j z_0^j$. The number of terms used depends on the degree of numerical precision we require, and the remainder $\sum_{j\geq N} p_j z^j$ can be well estimated by using the asymptotic expansion (7). More precisely, for large N,

$$\sum_{j\geq N} p_j z_0^j = \frac{6(z_0/\rho)^N}{\rho - z_0}\left(N + \frac{3\rho + 2z_0}{5(\rho - z_0)} + O\left(N^{-4}\right)\right). \tag{22}$$

Since $z_0 < \rho$, the right-hand side can be made arbitrarily small by choosing N sufficiently large so that the error introduced is under control.

Similarly, $U(Z) \approx U_M(Z) := \rho^{-1}\sum_{k<M} u_k(\log Z)Z^{k-2}$ for a sufficiently large M whose choice can be determined by the desired degree of precision and the upper bound (19). The errors introduced are thus bounded above by

$$\rho^{-1}\sum_{k\geq M} u_k(\tau_0)e^{(k-2)\tau_0} = O\left(M^{-2}B^M|1-\tau_0|^{M/6}e^{M\Re(\tau_0)}\right), \tag{23}$$

where $\tau_0 = \log(Z_0)$.

Note that if z_0 is too close to zero, then the remainder (22) for P decreases much faster than that (23) for U, and if z_0 is too close to ρ, then the converse is true. So the best choice for z_0 will be the one at which both remainders are asymptotically of the same order. For practical use, since p_n is easier to compute than u_k, we take $M = \beta N$ for some $\beta \in (0,1)$. Then we solve the equation

$$\left(\frac{z_0}{\rho}\right)^{1/\beta} = B\left|1 - \log\left(1 - \frac{z_0}{\rho}\right)\right|^{1/6}\left(1 - \frac{z_0}{\rho}\right), \tag{24}$$

(which has a unique real solution for $z_0/\rho \in (0,1)$ by monotonicity) to find a better z_0.

On the other hand, to compute u_k, we take the first entry of $\boldsymbol{\phi}_k$ in (17) and obtain, by integration by parts, the recurrence

$$\begin{aligned} u_k(\tau) = u_{k-1}(\tau) &+ \frac{1}{7}\int_0^\infty \left(9e^{-(k-6)x} - 16e^{-(k+1)x}\right)u_{k-1}(\tau - x)\,\mathrm{d}x \\ &+ \frac{1}{7}\int_0^\infty \left(e^{-(k-6)x} - e^{-(k+1)x}\right)\sum_{1\leq j<k} u_j(\tau - x)u_{k-j}(\tau - x)\,\mathrm{d}x, \end{aligned}$$

for $k \geq 7$. All u_k's are solvable recursively starting from the initial values (14).

We finally solve numerically the pair (ρ, c_6) from the two equations with $\rho \in (3,4)$

$$P_N(z_0) = U_M(Z_0) \quad \text{and} \quad P_N'(z_0) = -\frac{1}{\rho} U_M'(Z_0). \tag{25}$$

In this way, we obtain the numerical values of ρ and c_6 given in the Introduction (in Theorem 1 and the following paragraph).

Numerical evidence suggests that the series definition for $U(Z)$ and $U'(Z)$ are both convergent for $Z = 1$, which means that one might even use the two equations

$$U(1) = 1, \quad U'(1) = -\rho,$$

to solve for the pair (ρ, c_6). But the convergence is much slower than taking z_0 according to (24).

2.5 A quantity arising in phylogenetic trees

The following recurrence

$$q_n = \frac{2}{(n-1)^2} \sum_{1 \leq j < n} q_j q_{n-j} \qquad (n \geq 2),$$

with $q_1 = 1$ was introduced in Bryant et al. [9] in the course of analyzing the size of a maximum agreement subtree in two randomly chosen trees according to the Yule-Harding model, the context being similar to the equality of random BSTs. The quantity serves as an effective bound for the probability that the size of a common maximum agreement subtree exceeds a certain given value.

It is easy to check that $q_{n+1} = 2^n p_n$, where p_n is as above (see (1)). Thus, by (7), we obtain the asymptotic expansion

$$q_n = \left(\frac{\rho}{2}\right)^{-n} \left(3n - \frac{6}{5} + \frac{168}{3125} n^{-5} + \frac{336}{3125} n^{-6} + O\left(n^{-7}\right)\right).$$

where $\rho/2 = 1.57042\,87836\,01468\,47580\,40837\ldots.$

3 Probability of equality of random trees

The consideration of the equality of two random BSTs can be easily extended either to more random BSTs or to other variants of BSTs.

3.1 Equality of d random BSTs

We extend in this subsection the same psi-series analysis to d random BSTs, $d \geq 2$. Surprisingly, the resulting forms of the asymptotic expansions depend on the parity of d.

Recurrence. The random BST model is as introduced above. Let $p_n = p_n(d)$ denote the probability that d random BSTs, each independent of the others, are identical. More precisely, the probability that d random permutations whose corresponding BSTs are all the same. Then p_n satisfies the recurrence

$$p_n = n^{-d} \sum_{0\le j<n} p_j p_{n-1-j} \qquad (n \ge 1), \tag{26}$$

with $p_0 = 1$. Let $P(z) := \sum_{n\ge 0} p_n z^n$ be the generating function of p_n. Then $P(z)$ satisfies the nonlinear ODE of order d

$$\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^d P(z) = zP(z)^2, \tag{27}$$

with $p_0 = 1$ and the first $d-1$ values p_n for $1 \le n < d$ computable by the recurrence (26). Basic analytic properties of $P(z)$ such as the existence of a finite dominant singularity $\rho > 0$ can be derived as the case when $d = 2$.

Analytic continuation of $P(z)$. We extend the same continuity argument we used in Section 2.2 to prove that if $\rho_\theta e^{\theta \mathrm{i}}$ is a singularity of $P(z)$, where $\rho_\theta > 0$ and $\theta \ne 0$, then $\rho_\theta > \rho$; this proves that P is analytically continuable to a region outside its disk of convergence.

From the ODE (27), we define the following auxiliary functions: for $x \ge 0$,

- $\upsilon_0(x) := P(x)$, and $\upsilon_j(x) := x\upsilon_{j-1}'(x)$, $1 \le j \le d$;
- $\varpi(x) := |P(xe^{\theta\mathrm{i}})|$;
- $w_j(x) := \left|\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^j P(z)\Big|_{z=xe^{\theta\mathrm{i}}}\right|$, $1 \le j \le d$,

with $\varpi(0) = P(0)$, $\upsilon_j(0) = w_j(0)$, $1 \le j \le d$. Then $\upsilon_d'(x) = P(x)^2$ and we have the following inequalities: $w_j(x) \le \upsilon_j(x)$, $|w_j'(x)| \le w_{j+1}(x)/x$ for $1 \le j < d$, and $|w_d'(x)| \le \varpi(x)^2$.

We next show that we can find two points $0 < x_2 < x_1 < \rho$ such that

$$P(x_2) > \varpi(x_1), \ \upsilon_j(x_2) > w_j(x_1) \qquad (1 \le j \le d).$$

Again we show that $P(x_2+t) > \varpi(x_1+t)$ for $t \in J := [0, \min\{\rho_\theta - x_1, \rho - x_2\}]$; see Figure 5.

Assume on the contrary that there exists a smallest number $t_0 \in (0, \min\{\rho_\theta - x_1, \rho - x_2\})$ such that

$$P(x_2+t_0) = \varpi(x_1+t_0) \quad \text{and} \quad P(x_2+t) > \varpi(x_1+t), \qquad t \in [0, t_0). \tag{28}$$

Then $P'(x_2+t_0) \le \varpi'(x_1+t_0)$ implies that $(x_2+t_0)P'(x_2+t_0) \le (x_1+t_0)\varpi'(x_1+t_0)$ or, equivalently, $\upsilon_1(x_2+t_0) \le w_1(x_1+t_0)$. In view of this and the inequality $\upsilon_1(x_2) > w_1(x_1)$, we can choose a smallest number $t_1 \in (0, t_0]$ such that

$$\begin{aligned}
\upsilon_1(x_2+t_1) = w_1(x_1+t_1) \implies\ & \upsilon_1'(x_2+t_1) \le w_1'(x_1+t_1) \\
\implies\ & \upsilon_2(x_2+t_1) = (x_2+t_1)\upsilon_1'(x_2+t_1) \\
& \le (x_1+t_1)w_1'(x_1+t_1) \le w_2(x_1+t_1).
\end{aligned}$$

By repeating the same process, we can find a sequence of t_j's, $0 < t_d \leq t_{d-1} \leq \cdots \leq t_1 \leq t_0$ such that

$$\begin{aligned} v_j(x_2 + t_j) &= w_j(x_1 + t_j), \qquad (1 \leq j \leq d), \\ v_{j+1}(x_2 + t_j) &\leq w_{j+1}(x_1 + t_j), \qquad (1 \leq j < d). \end{aligned}$$

Finally, we have

$$\begin{aligned} v_d(x_2 + t_d) = w_d(x_1 + t_d) = w_d(x_1) &+ \int_0^{t_d} w_d'(x_1 + s)\,\mathrm{d}s \\ &< v_d(x_2) + \int_0^{t_d} \varpi(x_1 + s)^2\,\mathrm{d}s \\ &< v_d(x_2) + \int_0^{t_d} \underbrace{P(x_2 + s)^2}_{\text{by (28)}}\,\mathrm{d}s \\ &= v_d(x_2) + \int_0^{t_d} v_d'(x_2 + s)\,\mathrm{d}s = v_d(x_2 + t_d), \end{aligned}$$

a contradiction. Thus $P(x_2 + t) > \varpi(x_1 + t)$ for $t \in J := [0, \min\{\rho_\theta - x_1, \rho - x_2\}]$, and, consequently, $\rho_\theta \geq \rho + x_1 - x_2 > \rho$.

This completes the proof of the analytic continuation of P outside its disk of convergence.

The ARS Algorithm. As in the case of two random BSTs above, we begin with applying the ARS Algorithm and check first if there are pseudo-poles and incompatibility.

❶ Leading order analysis: This part is always easy for the problems we study in this paper and we obtain, by assuming $P(z) \sim c_0(1 - z/\rho)^{-\alpha}$ and by matching coefficients, $\alpha = d$ and $c_0 = (2d)!/(2\rho d!)$.

❷ Resonance analysis: On the other hand, by substituting the form (10) into (27) and by collecting the coefficient for the term $c_r(1 - z/\rho)^{r-2d}$ in the resulting expansion for (27), we obtain the polynomial characterizing all possible resonances

$$\begin{aligned} \Phi_d(r) &= \frac{(2d-1-r)!}{(d-1-r)!} - \frac{(2d)!}{d!} \\ &= \begin{cases} (r+1)\tilde{\Phi}_d(r), & d \text{ is odd;} \\ (r+1)(r-3d)\tilde{\Phi}_d(r), & d \text{ is even,} \end{cases} \qquad d \in \mathbb{N}, \end{aligned} \tag{29}$$

where $\tilde{\Phi}_d$ is a polynomial of even order and has no real zeros. We see that if d is odd, then there is no integer-valued resonance other than -1. Thus, the movable singularity ρ is a pole of order d. On the other hand, if d is even, then there exists an additional, unique, positive, integer-valued resonance $3d$ for each d.

❸ Incompatibility: We need only consider the case when d is even. The incompatibility of the resonance at $r = 3d$ is easily checked for each specific $d = 2, 4, 6, \ldots$, but a proof that $r = 3d$ leads to incompatibility for all such d is not obvious.

The case when d is odd. From the above quick check by the ARS algorithm, we see that the solution for the ODE (27) admits, by the Frobenius method, the Laurent series expansion

$$\rho P(z) = \frac{(2d)!}{2 \cdot d!} \left(Z^{-d} - \frac{(3d-2)(d-1)}{2(3d-1)} Z^{-d+1} + \sum_{2 \le j \le d} c_j Z^{j-d} \right) + \Xi(z),$$

where $\Xi(z) = \Xi_d(z)$ is analytic at ρ.

The case when d is even. Again, by the above procedure of ARS algorithm, we anticipate a psi-series expansion for $P(z)$ of the form

$$\rho P(z) = \sum_{j \ge 0} Z^{j-d} \sum_{0 \le \ell \le \lfloor j/3d \rfloor} c_{j,\ell} (\log Z)^{\ell}, \tag{30}$$

where the $c_{j,\ell}$'s are chosen so that the psi-series satisfies the ODE (27). In particular,

$$c_{0,0} = \frac{(2d)!}{2 \cdot d!} \quad \text{and} \quad c_{1,0} = -\frac{(3d-2)(d-1)(2d)!}{4(3d-1)d!}.$$

The justification of the psi-series on the right-hand side of (30) follows the same method of proof as that for two random BSTs; see Appendix A for more details.

In summary, we conclude the following asymptotic estimates, the drastic change of the error term according to the parity of d unveiling an additional surprise.

Theorem 2 *The probability that $d \ge 2$ randomly chosen BSTs are all equal satisfies*

$$p_n \rho^{n+1} = \frac{(2d-1)!}{(d-1)!^2} \left(n^{d-1} + \frac{(d-1)(2d-1)}{3d-1} n^{d-2} + \sum_{0 \le j \le d-3} C_j n^j \right) + \begin{cases} O((1-\varepsilon)^n), & \text{if } d \text{ is odd;} \\ K n^{-2d-1} + O\left(n^{-2d-2}\right), & \text{if } d \text{ is even,} \end{cases}$$

where $\varepsilon > 0$, the C_j's are constants, $\rho = \rho_d$ depends on d and K is a constant depending only on d.

More precise asymptotic expansions can be derived, but we content ourselves with the current form for simplicity of presentation.

3.2 Equality of two random m-ary search trees

The m-ary search trees are one of the natural extensions of BSTs to branching factors $m \ge 2$ beyond binary; see [32] for thorough discussions. Briefly, from a given sequence of numbers, we can construct an m-ary tree as follows. The first $m-1$ keys are stored in the root and sorted in increasing order, each of the remaining $n-m+1$ keys are then directed to one of the m subtrees of the root, corresponding to the m intervals split by the $m-1$ sorted keys, and are constructed recursively by the same procedure.

In the same vein, the probability q_n that two random m-ary search trees (under the same random permutation model) are identical is characterized by the following recurrence ($m \geq 2$)

$$q_n = \binom{n}{m-1}^{-2} \sum_{\substack{j_1+\cdots+j_m=n-m+1 \\ j_1,\dots,j_m \geq 0}} q_{j_1} \cdots q_{j_m} \qquad (n \geq m-1),$$

with the initial conditions $q_j = 1$, $0 \leq j \leq m-2$. The associated generating function $Q(z)$ then satisfies the following nonlinear ODE

$$\left(z^{m-1} Q^{(m-1)}(z)\right)^{(m-1)} = (m-1)!^2 Q^m(z), \tag{31}$$

with the initial conditions $Q(z) = 1 + z + \cdots + z^{m-2} + q_{m-1}z^{m-1} + \cdots$, where q_j, $m-1 \leq j \leq 2m-3$, are determined by the above recurrence. Analytic properties of Q, including singularity and analytic continuation, can be derived as above and are omitted here.

❶ Leading order analysis: The assumption $Q(z) \sim c_0(1-z/\rho)^{-\alpha}$ leads to $\alpha = -2$ and $\rho c_0 = \left((2m-1)!/(m-1)!^2\right)^{1/(m-1)}$.

❷ Resonance analysis: Assuming that $Q(z) \sim c_0(1-z/\rho)^{-2} + c_r(1-z/\rho)^{-2+r}$, we obtain the following algebraic equation characterizing all possible resonances

$$\prod_{2 \leq j < 2m} (r-j) - \frac{(2m)!}{2} = (r+1)(r-(2m+2))\phi_m(r) = 0,$$

where $\phi_m(r)$ is a polynomial of degree $2(m-2)$ and admits complex-conjugate zeros only. Thus we need to check if the ODE (31) is compatible at the resonance $r = 2m+2$.

❸ Incompatibility: Similar to the case of d random BSTs, the resonance $r = 2m+2$ is easily checked to be incompatible for each finite values of $m = 2, 3, \dots$, but it is far from being obvious to prove directly the incompatibility for all $m \geq 2$.

Let $\lambda_m := \left((2m-1)!/(m-1)!^2\right)^{1/(m-1)}$. Instead of proving the incompatibility of $r = 2m+2$ for all $m \geq 2$ and that ρ is not an essential singularity, we prove that the ODE (31) has the psi-series solution

$$\rho U(Z) = \sum_{j \geq 0} Z^{j-2} \sum_{0 \leq \ell \leq \lfloor j/(2m+2) \rfloor} c_{j,\ell} \log^{\ell} Z,$$

which converges absolutely in some cut-region $\mathscr{C}_\varepsilon$ (defined in (8)); see Appendix A for details. Then we connect $Q(z)$ and $U(Z)$ by the same arguments as those used above for two random BSTs. In this way, we obtain $c_{0,0} = \lambda_m$ and $c_{1,0} = -m\lambda_m/(2m+1)$.

From this expansion, we then derive the following approximation to q_n.

Theorem 3 *The probability $q_n = q_n(m)$ that two random m-ary search trees are equal satisfies the asymptotic approximation*

$$q_n = \lambda_m \rho^{-n-1} \left(n + \frac{m+1}{2m+1} + Kn^{-2m-1} + O\left(n^{-2m-2}\right)\right),$$

where $\rho = \rho_m$ and K both depend on m.

m	$q_n \sim$	λ_m
2	$\lambda_2\rho_2^{-n-1}\left(n+\frac{3}{5}+\frac{56}{3125}n^{-5}\right)$	6
3	$\lambda_3\rho_3^{-n-1}\left(n+\frac{4}{7}+\frac{6927696}{78236585}n^{-7}\right)$	$\sqrt{30}$
4	$\lambda_4\rho_4^{-n-1}\left(n+\frac{5}{9}+\frac{10419284224}{15568564095}n^{-9}\right)$	$\sqrt[3]{140}$
5	$\lambda_5\rho_5^{-n-1}\left(n+\frac{6}{11}+\frac{1526061507281984000}{194179984589469879}n^{-11}\right)$	$\sqrt[4]{630}$
6	$\lambda_6\rho_6^{-n-1}\left(n+\frac{7}{13}+\frac{132275788517112977050000}{942913507718961369877}n^{-13}\right)$	$\sqrt[5]{2772}$

Table 2: *The asymptotic approximation to the probability that two random m-ary search trees are equal for $m = 2, \dots, 6$. All smaller order terms are omitted.*

As for BSTs, the consideration can be extended to choose $d \geq 2$ random m-ary search trees, and the resonance equation is given by

$$\prod_{0\le j<d(m-1)} (d-r+j) - \frac{m(dm-1)!}{(d-1)!} = \frac{\Gamma(d-r+d(m-1))}{\Gamma(d-r)} - \frac{m(dm-1)!}{(d-1)!}.$$

We then deduce that this equation has no positive integral resonance when m is even and d is odd, and has the positive resonance $d(m+1)$ for all other cases with $d, m \geq 2$. Our approach can be applied and we obtain an asymptotic approximation to the probability that d random m-ary search trees are equal, *the error terms beyond the constant term being either exponentially small when m is even and d is odd or of order $\asymp n^{-dm-1}$ for all the remaining meaningful cases.*

3.3 Equality of two random fringe-balanced BSTs

Median-of-$(2t+1)$ (or fringe-balanced) BSTs represent yet another class of extensions of BSTs. The idea is, instead of placing the first element in the given sequence at the root, which may result in a less balanced binary tree, we take a small sample of size $2t+1$ and use the median of this sample as the root element, which then partitions the remaining elements as in the construction of BSTs, where $t \geq 0$. This simple balancing scheme has turned out to be useful for small t, notably for the corresponding quicksort algorithm. Note that the original BST corresponds to $t = 0$.

For the probability model, assume, as in random BSTs, that we are given a random permutation of n elements; then we construct the corresponding median-of-$(2t+1)$ BST, which is called a *random median-of-$(2t+1)$ BST*.

Let now $f_n = f_n(t)$ denote the probability that two randomly chosen permutations lead to the same median-of-$(2t+1)$ BST. Then f_n satisfies the recurrence

$$f_n = \sum_{t\le j\le n-1-t} \frac{\binom{j}{t}^2\binom{n-1-j}{t}^2}{\binom{n}{2t+1}^2} f_j f_{n-1-j} \qquad (n \geq 2t+1), \tag{32}$$

with the initial conditions $f_n = 1$ for $0 \leq n \leq 2t$.

Let $F(z) := \sum_{n\ge0} f_n z^n$ denote the generating function of f_n. Then $F(z)$ satisfies the ODE

$$\left(z^{2t+1}F^{(2t+1)}(z)\right)^{(2t+1)} = \frac{(2t+1)!^2}{t!^4}\left(\left(z^t F^{(t)}\right)^{(t)}(z)\right)^2, \tag{33}$$

with the initial conditions $F^{(j)}(0) = j!$, $0 \le j \le 2t$, and f_j, $2t+1 \le j \le 4t+1$, determined by the recurrence (32). Analytic properties (movable singularity and analytic continuation) are derived by the same arguments we used for random BSTs; details are less interesting and omitted here.

❶ Leading order analysis: With the simple form $F(z) \sim c_0(1 - z/\rho)^{-\alpha}$, we obtain $\alpha = 2$ and $\rho c_0 = (4t+3)!\, t!^4/(2t+1)!^4$ for each $t \ge 0$.

❷ Resonance analysis: Again, assuming that $F(z) \sim c_0(1 - z/\rho)^{-2} + c_r(1 - z/\rho)^{-2+r}$, we obtain the resonance equation

$$\Phi_t(r) = \left(\prod_{2 \le j \le 2t+1} (r - j)\right)\left(\prod_{2t+2 \le j \le 4t+3} (r - j) - 2 \prod_{2t+2 \le j \le 4t+3} j\right),$$

which can be factored into the form

$$(r+1)(r - 6t - 6)\, \tilde{\Phi}_t(r)(r-2)\cdots(r - 2t - 1),$$

where $\tilde{\Phi}_t(r)$ has only complex conjugate zeros since the factor

$$\begin{aligned}&(r - 2t - 2)\cdots(r - 4t - 3) - 2(2t+2)\cdots(4t+3)\\ &= (r - 2t - 2)\cdots(r - 4t - 3) - (2t+3)\cdots(4t+4)\end{aligned}$$

never vanishes for $r \in \mathbb{R} \setminus \{-1, 6t+6\}$. Thus we get yet another new pattern for the least positive integer-valued resonance

$$r = \begin{cases} 6, & t = 0,\\ 2, & t \ge 1.\end{cases}$$

❸ Incompatibility: Since $t = 0$ has already been addressed in Section 2, we focus on $t \ge 1$, which has the constant resonance $r = 2$. A direct check of the incompatibility is possible for $r = 2$ and $t \ge 1$; see Appendix B.

The same psi-series method applies and we obtain for $t \ge 1$

$$\begin{aligned}\rho F(z) = \frac{(4t+3)!t!^4}{(2t+1)!^4}&\left(Z^{-2} - \frac{2(t+1)^2}{6t+5} Z^{-1} + \frac{\left(22t^2 + 35t + 14\right)(t+1)^2 t}{(7t+6)(6t+5)^2} \log Z\right)\\ &+ O\left(|Z||\log Z|\right).\end{aligned}$$

Theorem 4 *The probability f_n that two random median-of-$(2t+1)$ BSTs are equal satisfies the asymptotic approximation*

$$\begin{aligned}f_n = \frac{(4t+3)!t!^4}{(2t+1)!^4}\rho^{-n-1}&\left(n + \frac{3 + 2t - 2t^2}{6t+5} - \frac{\left(22t^2 + 35t + 14\right)(t+1)^2 t}{(7t+6)(6t+5)^2} n^{-1}\right)\\ &+ O\left(\rho^{-n} n^{-2}\right),\end{aligned}$$

for $t \ge 1$, where $\rho = \rho_t$ is an effectively computable constant.

Note that the expansion also holds when $t = 0$ but the O-term becomes $O(\rho^{-n} n^{-5})$; see (7). Also more terms can be computed by the same procedure.

4 Asymptotics of some quadratic recurrences of Faltung type

In addition to the equality of random trees, the asymptotics of quadratic convolution recurrences provides another rich source of nonlinear recurrences and differential equations of the same type as those analyzed in previous sections.

4.1 Partial match queries in random quadtrees

We consider first in this section the cost of partial match queries in random two-dimensional quadtrees. The expected cost was first analyzed in Flajolet et al. [19] (see also [10]) and the limit law derived in Neininger and Rüschendorf [37]; see also [8, 14] for recent progresses along this direction.

Let $v := (\sqrt{17}-3)/2$. Then the cost of a random partial match query in a random two-dimensional quadtree of n nodes tends (under a purely *idealized* model where randomness is preserved for all subtrees), after normalized by n^v, to a limit law X whose moments satisfy (see [37])

$$\mathbb{E}(X^m) = \frac{x_m}{\Gamma(mv+1)},$$

where $x_1 := \Gamma(2v+2)/(2\Gamma(v+1)^2)$ and

$$x_m = \frac{2}{v(m-1)((m+1)v+3)} \sum_{1\le j<m} \binom{m}{j} x_j x_{m-j} \qquad (m \ge 2).$$

Then the generating function $X(z) := 1 + \sum_{m\ge1} x_m z^m/m!$ satisfies the differential equation

$$v^2 z^2 X''(z) + 2zX'(z) + 2X(z) = 2X^2(z), \tag{34}$$

with the initial conditions $X(0) = 1$ and $X'(0) = x_1$.

The psi-series method we use above can be readily applied and we obtain the resonance $r = 6$ and

$$\begin{aligned} X(z) = 3v^2 Z^{-2} + \frac{6}{5}(9v-5)Z^{-1} + \sum_{2\le j\le 7} c_j Z^{j-2} + \frac{117(39v+139)}{43750} Z^4 \log Z \\ + \frac{468(153v+545)}{109375} Z^5 \log Z + O\left(|Z|^6 |\log Z|\right), \end{aligned} \tag{35}$$

where the c_j's are unimportant constants. By singularity analysis (see [21]), we then conclude the following asymptotic approximation to $x_m/m!$.

Theorem 5 *The m-th moment of X satisfies for large m*

$$\begin{aligned} \mathbb{E}(X^m) = \frac{m!\rho^{-m}}{\Gamma(mv+1)} \Bigg(3v^2 m + \frac{9}{5}v - \frac{1404(39v+139)}{2185\, m^5} \\ + \frac{8424(139v+495)}{21875\, m^6} + O\left(m^{-7}\right) \Bigg), \end{aligned}$$

where $\rho \approx 1.37649\,44410\,57156\,25755\ldots$.

We omit all details as they are very similar to the case of the equality of two random BSTs.

An interesting implication of our psi-series analysis is that we can derive an asymptotic expansion for the moment generating function of X

$$\mathbb{E}\left(e^{Xz}\right) = e^{(z/\rho)^{1/v}} \left(3\left(\frac{z}{\rho}\right)^{1/v} + \frac{9}{5} - \frac{22464}{21875}\left(\frac{z}{\rho}\right)^{-5/v} + O(|z|^{-6/v}) \right), \tag{36}$$

as $|z| \to \infty$ in the sector $|\arg(z)| \leq (v-\varepsilon)\pi/2$. This is proved by the integral representation

$$\mathbb{E}\left(e^{Xz}\right) = \frac{1}{2\pi \mathrm{i}} \int_{\mathscr{H}} e^s s^{-1} X(z/s^v)\, \mathrm{d}s,$$

for a suitable Hankel-type contour, and standard analysis; see Appendix C for a proof. Such an expansion for the moment generating function is unusual in the probability literature and implies in turn that

$$-\log \mathbb{P}(X > t) \sim (1-v)v^{v/(1-v)}(\rho t)^{1/(1-v)},$$

for large t, by an application of a Tauberian argument; see Section 4.12 of Bingham et al. [6].

Note that the transformations $z = \xi^{-v}$ and $X(z) = 2\xi \hat{X}(\xi)$ bring the ODE (34) to the standard form of the so-called *Emden's equation*

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\hat{X}(\xi) = \xi^{-1}\hat{X}^2(\xi).$$

But it is still not exactly solvable; see [28, § 12.4] or [36, § 2.3].

4.2 Partial match queries in random relaxed k-d trees

In a similar setting, the cost of a random partial match query in a random relaxed k-d trees (see [16]) tends, after proper normalization, to the limit law Y whose moments satisfy (see [35])

$$\mathbb{E}(Y^m) = \frac{y_m}{\Gamma(m\beta+1)},$$

where $\beta := (-1+\sqrt{9-8s/k})/2$ (s out of the k coordinates in the query pattern is specified, the other $k-s$ being "don't-cares"), and

$$y_m = \frac{\beta+1}{(m-1)((m+1)\beta+1)} \sum_{1\leq j<m} \binom{m}{j} (j\beta+1) y_j y_{m-j} \qquad (m \geq 2),$$

with

$$y_1 = \frac{2\Gamma(2\beta+2)}{\beta(\beta+1)^2(2\beta+1)\Gamma^3(\beta+1)}.$$

It follows that the generating function $Y(z) := 1 + \sum_{m\geq 1} y_m z^m/m!$ satisfies the nonlinear differential equation

$$\beta z^2 Y''(z) + (\beta+1)^2 z Y'(z) + (\beta+1)Y(z) = (\beta+1)Y^2(z) + \beta(\beta+1)zY'(z)Y(z),$$

with the initial conditions $Y(0) = 1$ and $Y'(0) = y_1$.

The psi-series method applies with a resonance at $r = 2$ and we obtain the expansion

$$Y(z) = \frac{2}{\beta+1} Z^{-1} + \frac{\beta-1}{\beta} + c_2 Z + \frac{2(\beta-1)(\beta+2)}{3\beta^2(\beta+1)} Z\log Z + c_3 Z^2 + \frac{(\beta-1)(\beta+2)(\beta+3)}{3\beta^3(\beta+1)} Z^2 \log Z + c_4 Z^3 + O\left(|Z|^3 |\log Z|\right),$$

from which we deduce an asymptotic approximation to higher order moments of Y.

Theorem 6 *The m-th moment of the limit law Y satisfies*

$$\mathbb{E}(Y^m) = \frac{2m!\rho^{-m}}{(\beta+1)\Gamma(m\beta+1)} \left(1 + \frac{(\beta-1)(\beta+2)}{3\beta^2 m^2} - \frac{(\beta-1)(\beta+2)}{\beta^3 m^3} + O\left(m^{-4}\log m\right)\right),$$

as $m \to \infty$, where ρ depends on β.

Implications of the expansion for $Y(z)$ can be derived as those for X.

4.3 Recursive partition structures.

In the context of recursive interval splitting, Gnedin and Yakubovich [24] derived the following recurrence relation for the m-th moment h_m of certain limit law W (satisfying a fixed-point equation with Dirichlet distribution as prefactors)

$$h_m = \frac{\Gamma(d+\omega)}{\Gamma(\omega)^2\Gamma(m\lambda+d+\omega)} \sum_{0\le j\le m} \binom{m}{j} \Gamma(j\lambda+\omega)\Gamma((m-j)\lambda+\omega) h_j\, h_{m-j}, \qquad (37)$$

for $m \ge 2$ with $h_0 = h_1 = 1$, where $\lambda, \omega > 0$ (λ is referred to as the *Malthusian exponent*) and $d = 2, 3, \ldots$

The case when $d = 2$. Consider first the simplest case when $d = 2$. In this case, the generating function

$$h(z) := \sum_{m\ge 0} \frac{h_m \Gamma(m\lambda+\omega)}{m!\Gamma(\omega)} z^m, \qquad (38)$$

satisfies the ODE (using the relation $(\lambda+\omega)(\lambda+\omega+1) = 2\omega(\omega+1)$)

$$v z^2 h''(z) + z h'(z) + h(z) = h^2(z),$$

which is exactly of the type of problems we have been examining in this paper (compare (34)), where for simplicity

$$v := \frac{\lambda^2}{\omega(\omega+1)}.$$

For this ODE, we can apply the psi-series method and obtain ($Z = 1 - z/\rho$)

$$h(z) = 6vZ^{-2} - \frac{6}{5}(6v-1)Z^{-1} + \sum_{2\le j\le 6} c_j Z^{j-2} + KZ^4 \log Z + O\left(|Z|^5 |\log Z|\right),$$

where $K := (v-1)^2(v-6)(6v-1)(2v+3)(3v+2)/(43750v^5)$. Consequently, we deduce the asymptotic expansion for the moments of W

$$h_m = \frac{6m!\Gamma(\omega)\rho^{-m}}{\Gamma(m\lambda+\omega)}\left(vm - \frac{v-1}{5} - 4Km^{-5} + O\left(m^{-6}\right)\right),$$

for large m.

The case when $d \geq 2$. From the recurrence (37), the generating function $h(y)$ (defined as in (38)) satisfies the ODE

$$y^{1-\omega}\frac{\mathrm{d}^d}{\mathrm{d}y^d}\left(h(y^\lambda)y^{d+\omega-1}\right) = \omega^{\overline{d}}h(y^\lambda)^2,$$

where $\omega^{\overline{d}} = \omega\cdots(\omega+d-1)$ denotes the rising factorial; see [24]. The ODE is however less manageable. We rewrite it as follows. Let $z = y^\lambda$ and $H(z) = z^\kappa h(z)$, where $\kappa := (d+\omega-1)/\lambda$. Note that the Malthusian exponent λ satisfies the relation

$$\frac{\omega^{\overline{d}}}{(\lambda+\omega)^{\overline{d}}} = \frac{1}{2}.$$

Then the function $H(z)$ satisfies the ODE

$$\lambda\theta(\lambda\theta-1)\cdots(\lambda\theta-d+1)H(z) = z^{-\kappa}\omega^{\overline{d}}H(z)^2, \tag{39}$$

where the differential operator θ is defined as $\theta := z(\mathrm{d}/\mathrm{d}z)$.

The leading order analysis and the resonance analysis give the dominant exponent $-d$ and the resonance equation is exactly the same as (29) for all $d \geq 2$, namely, $(d-r)^{\overline{d}} - (d+1)^{\overline{d}}$. It follows that we have the same asymptotic pattern for H as the case of d random BSTs.

The case when d is odd. The movable singularity ρ is a pole of order d and the solution $H(z)$ admits the Laurent expansion

$$\rho^{-\kappa}H(z) = \frac{(2d)!\lambda^d}{2\cdot d!\omega^{\overline{d}}}\sum_{0\leq j\leq d} c_j Z^{j-d} + \Xi_1(z),$$

where

$$c_0 = 1, \quad c_1 = -\frac{d}{2} - \frac{(4d-2)\omega + (d-1)(5d-2)}{2(3d-1)\lambda}, \tag{40}$$

and $\Xi_1(z)$ is an analytic function at $z = \rho$.

The case when d is even. In this case, since the resonance equation (29) possesses the unique positive integral resonance $3d$, we see that $z = \rho$ is a pseudo-pole and the psi-series solution to (39) has the form

$$\begin{aligned}\rho^{-\kappa}H(z) &= \sum_{j\geq 0} Z^{j-d}\sum_{0\leq\ell\leq\lfloor j/3d\rfloor} c_{j,\ell}(\log Z)^\ell \\ &= \frac{(2d)!\lambda^d}{2\cdot d!\omega^{\overline{d}}}\sum_{0\leq j\leq 3d} c_j Z^{j-d} + KZ^{2d}\log Z + O\left(|Z|^{2d+1}|\log Z|\right),\end{aligned}$$

where, in particular, c_0 and c_1 are given as in (40), and K is a constant dependent on λ and ω.

Expansions for h. It is not difficult to verify that $h(z)$ and $H(z)$ have the same dominant singularity ρ, dominant exponent $-d$, and the dominant resonance $3d$. Now by the relation $h(z) = (1-Z)^{-\kappa}\rho^{-\kappa} H(z)$, we obtain

$$h(z) = \frac{(2d)!\lambda^d}{2\cdot d!\omega^{\overline{d}}} \times \begin{cases} \sum_{0\le j\le d} c'_j Z^{j-d} + \Xi_2(z), & \text{if } d \text{ is odd;} \\ \sum_{0\le j\le 3d} c'_j Z^{j-d} + K' Z^{2d}\log Z \\ \quad +O\left(|Z|^{2d+1}|\log Z|\right) & , \text{ if } d \text{ is even,} \end{cases}$$

where $c'_0 = 1$,

$$c'_1 = \frac{d}{2}\left(\frac{d+2\omega-1}{(3d-1)\lambda} - 1\right),$$

and Ξ_2 is analytic at $z = \rho$.

Asymptotics of the moments. From the expansions we derived and a similar analysis as for d random BSTs, we conclude the following asymptotic approximations to the limit law W.

Theorem 7 *The m-th moment h_m of W satisfies*

$$h_m\rho^m = \frac{(2d)!\Gamma(\omega)^2\lambda^d m!}{2\cdot d!(d-1)!\Gamma(\omega+d)\Gamma(m\lambda+\omega)} \sum_{0\le j\le d} C_j m^{d-1-j}$$
$$+ \begin{cases} O((1-\varepsilon)^m), & \text{if } d \text{ is odd;} \\ C m^{-2d-1} + O\left(m^{-2d-2}\right), & \text{if } d \text{ is even,} \end{cases}$$

for large m, where $\varepsilon \in (0,1)$, ρ, C and the C_j's are constants depending on d, λ, ω. In particular, $C_0 = 1$ and

$$C_1 = \binom{d}{2}\frac{d+2\omega-1}{(3d-1)\lambda}.$$

4.4 An Ansatz solution in Boltzmann equations

The following sequence t_n arose in the analysis (see [3]) of exact solutions of the Tjon-Wu representation of Boltzmann equations (which represent the major cornerstone of kinetic theory in statistical mechanics). Let ν be a positive integer. The sequence t_n is defined recursively as

$$\left(\frac{\nu(\nu+1)}{\nu+2}n(n-1) - (n+1)\right)t_n = -\sum_{0\le j\le n} t_j t_{n-j} \qquad (n \ge 2), \tag{41}$$

with $t_0 = t_1 = 1$. This recurrence translates into the following ODE for the generating function $T(z) := \sum_{n\ge 0} t_n z^n$

$$\frac{\nu(\nu+1)}{\nu+2} z^2 T''(z) - zT'(z) - T(z)\left(1 - T(z)\right) = 0, \tag{42}$$

with the initial conditions $T(0) = T'(0) = 1$.

Straightforward computations as above give -2 as the dominant exponent for the dominant term of $T(z)$ and $(r+1)(r-6)$ as the resonance equation for each $\nu = 1, 2, \ldots$. Interestingly, for the resonance $r = 6$, the two special cases $\nu = 1, 2$ do not lead to incompatible system of equations, in contrast to all higher values of ν. This is very different from the cases we have been dealing with up to now. According to the ARS method, the cases when $\nu = 1, 2$ admit the *Painlevé property* [11, §1.2, Definition 1.1] and have solutions in terms of Laurent expansion with two free parameters; in other words, they are *integrable*, and we will derive closed-form solutions for them. The remaining cases when $\nu \geq 3$ have psi-series solutions.

Exactly solvable (integrable) case: $\nu = 1$. We start with the case $\nu = 1$. Consider the transformations $T(z) = 1 - \zeta V(\zeta)$ and $z = -\zeta$. Note that, by this transform, the coefficients $[\zeta^n]V(\zeta)$ are positive and the transformed ODE (after multiplying $V'(\zeta)$) becomes

$$\frac{1}{3}\zeta^2 \frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\zeta V'(\zeta)^2 - V(\zeta)^3\right) = 0 \quad \text{or} \quad \sqrt{\zeta}V'(\zeta) = \sqrt{V(\zeta)^3 - 1},$$

with $V(0) = 1$. Also $V'(0) = 3$. Then V is solved implicitly as

$$2\sqrt{\zeta} = \int_1^{V(\zeta)} \frac{\mathrm{d}x}{\sqrt{x^3 - 1}}. \tag{43}$$

Let

$$2\sqrt{\zeta_\infty} = \int_1^{\infty} \frac{\mathrm{d}x}{\sqrt{x^3 - 1}} \approx 2.42865\,06478\,87581\,61181\ldots,$$

or $\zeta_\infty \approx 1.47458\,59923\,71192\,48035\ldots$. Obviously $V(\zeta) \to \infty$ as $\zeta \to \zeta_\infty$. Let $\Delta := 2(\sqrt{\zeta_\infty} - \sqrt{\zeta})$. Then (43) can be written as

$$\Delta = \int_{V(\zeta)}^{\infty} \frac{\mathrm{d}x}{\sqrt{x^3 - 1}}.$$

Since $V(\zeta) \to \infty$ as $\zeta \to \zeta_\infty$, we deduce that

$$\Delta = 2V(\zeta)^{-1/2} + \frac{1}{6}V(\zeta)^{-7/2} + \frac{3}{52}V(\zeta)^{-13/2} + \frac{5}{152}V(\zeta)^{-19/2} + \text{smaller order terms}.$$

Consequently, by inverting the series (justified by analyticity and standard arguments), we obtain

$$V(\zeta) = 4\Delta^{-2} + \frac{\Delta^4}{112} + \frac{\Delta^{10}}{652288} + \frac{\Delta^{16}}{5552275456} + \text{smaller order terms}.$$

Finally, let $\rho := -\zeta_\infty$ and let $[z^n]f(z)$ denote the coefficient of z^n in the Taylor expansion of f. We obtain

$$\begin{aligned} t_n = [z^n]T(z) &= (-1)^{n-1}[\zeta^{n-1}]V(\zeta) \\ &\sim 8(-1)^{n-1}[y^{2n-2}]\left(2\sqrt{\zeta_\infty} - 2y\right)^{-2} \\ &= 2(-1)^{n-1}(2n-1)|\rho|^{-n}, \end{aligned}$$

the errors omitted being exponentially smaller.

Exactly solvable (integrable) case: $\nu = 2$. The case when $\nu = 2$ is similar. We now adopt the transformations $T(z) = 1 - \zeta^2 L(\zeta)$ and $z = -\zeta^3$. Then the ODE (41) becomes

$$L''(\zeta) - 6L(\zeta)^2 = 0 \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{1}{2}L'(\zeta)^2 - 2L(\zeta)^3\right) = 0,$$

with the initial values $L(0) = 0$ and $L'(0) = 1$. Thus, the solution is given by

$$\zeta = \int_0^{L(\zeta)} \frac{\mathrm{d}x}{\sqrt{1+4x^3}}. \tag{44}$$

Let ζ_∞ denote the dominant singularity of $L(\zeta)$. Then

$$\zeta_\infty = \int_0^\infty \frac{\mathrm{d}x}{\sqrt{4x^3+1}} = \frac{2^{1/3}}{6}\mathrm{Beta}\left(\frac{1}{6}, \frac{1}{3}\right) \approx 1.76663\,87502\,85449\,95731\dots.$$

Thus the dominant singularity of $T(z)$ when $\nu = 2$ is

$$\rho = -\zeta_\infty^3 = -\frac{1}{108}\mathrm{Beta}\left(\frac{1}{6}, \frac{1}{3}\right)^3 \approx -5.51370\,15767\,10567\,75506\dots.$$

By (44) and the same procedure as above, we have ($\Delta := \zeta_\infty - \zeta$)

$$L(\zeta) = \Delta^{-2} - \frac{\Delta^4}{28} + \frac{\Delta^{10}}{10192} - \frac{\Delta^{16}}{5422144} + \frac{3\Delta^{22}}{9868302080} - \text{smaller order terms}.$$

Accordingly,

$$\begin{aligned} t_n = [z^n]T(z) &= 3(-1)^{n-1}[\zeta^{3n-2}]L(\zeta) \\ &\sim 3(-1)^{n-1}(3n-1)|\rho|^{-n}. \end{aligned}$$

Note that we can use the transforms $z = \zeta^2$ and $T(z) = 1 - V(\zeta)\zeta^2$ to convert the ODE for $\nu = 1$ to a ODE of same type (differing only by a constant) as the case for $\nu = 2$. Also both solutions can be expressed in terms of Weierstrass $\wp$ functions.

The remaining cases: $\nu \geq 3$. Unlike the preceding two cases, the remaining ν's no longer lead to ODEs that are solvable by *quadrature*[1]. Due to incompatibility, we apply instead the psi-series method. Because of the negative sign on the right-hand side of (41), we consider the transform $z = -\zeta$ and $T(z) = 1 - \zeta V(\zeta)$. Then

$$t_n = [z^n]\,T(z) = (-1)^{n-1}\left[\zeta^{n-1}\right]V(\zeta),$$

and (42) is translated into

$$\frac{\nu(\nu+1)}{\nu+2}\zeta V''(\zeta) + \frac{2\nu^2+\nu-2}{\nu+2}V'(\zeta) - V(\zeta)^2 = 0.$$

[1] An ODE is said to be *solvable* by quadrature if its solution can be expressed in terms of one or more integrations.

Let now $Z = 1 - \zeta/\rho$, where $\rho > 0$ is the dominant singularity of V (having all Taylor coefficients positive). Then we deduce the psi-series expansion for V

$$\rho V(\zeta) = \frac{6\nu(\nu+1)}{(\nu+2)Z^2} - \frac{6(\nu^2+2\nu+2)}{5(\nu+2)Z} + \sum_{0\le j\le 5} c_j Z^j + KZ^4\log Z + O\left(|Z|^5|\log Z|\right),$$

where

$$K := -\frac{(\nu-1)(\nu-2)(\nu+3)(\nu+4)(2\nu+1)(2\nu+3)(3\nu+2)(3\nu+4)\left(\nu^2+2\nu+2\right)^2}{43750\nu^5(\nu+1)^5(\nu+2)}.$$

This, together with the approximations we derived for t_n in the two cases $\nu = 1, 2$, implies the following asymptotics of t_n. Note that $K = 0$ when $\nu = 1, 2$.

Theorem 8 *The sequence t_n satisfies the asymptotic expansion*

$$(-1)^{n-1}t_n = \rho^{-n}\left(\frac{6\nu(\nu+1)}{\nu+2}n - \frac{6(\nu^2+2\nu+2)}{5(\nu+2)} + \begin{cases} O((1-\varepsilon)^n), & \text{if } \nu = 1,2; \\ 24Kn^{-5} + O(n^{-6}), & \text{if } \nu \ge 3. \end{cases}\right)$$

5 Conclusions

In the literature, the first use of the psi-series dates back to at least Jakob Horn's work (see [29]) in the late 19th century. Psi-series have been introduced and applied in many diverse subject areas over the years. Through the concrete examples we studied in this paper (see Table 3 for a table summary), we see that the psi-series method is a powerful approach for several problems in applied probability and analysis of algorithms. It is especially useful for handling nonlinear ODEs (mostly from quadratic convolution recurrences) and leads to some surprising results, notably asymptotic expansions with missing terms. The procedure we adapted and improved from Hille's for proving the absolute convergence of psi-series is of certain generality and can be applied to other problems of similar nature.

Another feature of the recurrences we studied in this paper is that they are very sensible to small variations, the example of d random BSTs being typical. Note first that the recurrence (26) with $d = 0$ yields the well-known Catalan numbers and the case $d = 1$ gives rise to the trivial sequence $p_n = 1$. The case $d = 1$ in a more general form was studied by Wright [40]; see also Cooper [12] for a study of p_n for real $d \ge 0$.

We now compare the recurrence (26) with the following one by defining $p_1 = 1$ and

$$p_n = n^{-d}\sum_{1\le j\le n-1} p_j p_{n-j} \qquad (n \ge 2).$$

While the case $d = 0$ still yields the Catalan numbers with their generating function satisfying

$$P(z) - z = P^2(z),$$

the case $d = 1$ becomes a nonlinear differential equation of Riccati type

$$zP'(z) - z = P^2(z), \qquad P(0) = 0,$$

which can still be explicitly solved $P(z) = z^{1/2}J_1(2z^{1/2})/J_0(2z^{1/2})$, where $J_\nu(z)$'s are Bessel functions (see [30]). The case $d = 2$ is again of Emden-Fowler type and can be solved asymptotically by psi-series method as well as the remaining cases $d \ge 3$.

See [12, 20, 22, 31, 39, 40] and the references therein for some quadratic recurrences of the above "Faltung" type. More examples can be found in the recent papers [4, 5].

DE	psi-series (resonance)	Frobenius	Section
$zP'' + P' = P^2$	$r = 6$		2
$(z\mathbb{D})^d P = zP^2$	d even $r = 3d$	d odd	3.1
$(z^{m-1}Q^{(m-1)})^{(m-1)} = (m-1)!^2 Q^m$	$r = 2m + 2$		3.2
$(z^{m-1}\mathbb{D}^{m-1})^d Q = (m-1)!^d z^{m-1} Q^m$	$(m, d) \neq$ (even,odd) $r = d(m+1)$	$(m, d) =$ (even,odd)	3.2
$(z^{2t+1}F^{(2t+1)})^{(2t+1)} = \frac{(2t+1)!^2}{t!^4}\left(\left(z^t F^{(t)}\right)^{(t)}\right)^2$	$r = 6$ $(t = 0)$ $r = 2$ $(t \geq 1)$		3.3
$v^2 z^2 X'' + 2zX' + 2X = 2X^2$	$r = 6$		4.1
$\frac{\beta z^2}{\beta+1} Y'' + (\beta + 1)zY' + Y = Y^2 + \beta zYY'$	$r = 2$		4.2
$\lambda\theta \cdots (\lambda\theta - d + 1)H = z^{-\kappa}\omega^{\overline{d}} H^2$	d even $r = 3d$	d odd	4.3
$\frac{\nu(\nu+1)}{\nu+2} z^2 T'' - zT' = T(1 - T)$	$\nu \geq 3$ $r = 6$	$\nu = 1, 2$	4.4

Table 3: *All nonlinear DEs studied in this paper; the resonance is specially marked when the psi-series method applies.*

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A Proof of the absolute convergence of psi-series

In this Appendix, we group the proof details for the absolute convergence of the psi-series arising in the three cases: d random BSTs, two random m-ary search trees, and two random median-of-$(2t+1)$ BSTs. We first describe briefly the general proof pattern and then provide more details for each case.

Our proof begins with rewriting the original ODE in z into a system of linear ODEs in $Z = 1 - z/\rho$ of the form

$$\frac{\mathrm{d}}{\mathrm{d}Z}\mathbf{U}(Z) = \boldsymbol{\mathcal{X}}(Z, \mathbf{U}), \quad \mathbf{U}(Z) = \begin{pmatrix} U_1(Z) \\ \vdots \\ U_s(Z) \end{pmatrix}, \tag{A.1}$$

where $s \in \{d, 2(m-1), 4t+2\}$, $\boldsymbol{\mathcal{X}} : \mathbb{C}^{s+1} \mapsto \mathbb{C}^s$, and $U_j(Z) = \sum_{k\geq 0} u_k^{[j]}(\tau) Z^{-\alpha+k-j+1}$ with α the leading order and $\tau = \log Z$. Then we derive the infinite system of linear ODEs satisfied by the $u_k^{[j]}$'s

$$\dot{\boldsymbol{\phi}}_k + \mathbf{A}_k \boldsymbol{\phi}_k = \mathbf{g}_k, \quad \boldsymbol{\phi}_k = \begin{pmatrix} u_k^{[1]} \\ \vdots \\ u_k^{[s]} \end{pmatrix}, \tag{A.2}$$

where $\mathbf{A}_k = k\mathbf{I}_{s\times s} - \mathbf{M}$ and $\mathbf{M} \in \mathbb{C}^{s\times s}$ are $s \times s$ matrices.

Relying on such an infinite system, we derive an upper bound for all $u_k^{[j]}$ (in particular, for $u_k^{[1]}$) of the form

$$\max_{1\leq j\leq s} \left| u_k^{[j]}(\tau) \right| \leq \frac{C_0}{(k+1)^2} B^k |1-\tau|^{k/c(s)}, \tag{A.3}$$

for $\tau \in \mathscr{T} := \left\{\tau \ : \ B|1-\tau|^{1/c(s)} e^{\Re(\tau)} < 1, \Re(\tau) < 0, -\pi < \theta \leq \pi\right\}$ (see Figure 7 on page 13), where $B > 0$ is a constant and $c(s)$ depends on the problem in question. The absolute convergence can then be justified as in the case of two random BSTs; see Proposition 2.

An additional common and interesting feature this approach brings is that *the resonance equation equals to* $\det(r\mathbf{I}_{s\times s} - \mathbf{M})$. We will explain this in more detail.

The following relations are useful in converting our ODEs in z into those in Z ($\mathbb{D} = \mathrm{d}/\mathrm{d}z$).

$$z = \rho(1-Z), \quad z\mathbb{D} = -(1-Z)\frac{\mathrm{d}}{\mathrm{d}Z}, \quad z^j \mathbb{D}^j = (-1)^j (1-Z)^j \frac{\mathrm{d}^j}{\mathrm{d}Z^j}.$$

Equality of d random BSTs. The linear system (A.1) specializing to (27) is

$$\begin{cases} U_j'(Z) = (1-Z)^{-1} U_{j+1}(Z), & 1 \leq j < d, \\ U_d'(Z) = (-1)^d \rho U_1(Z)^2. \end{cases}$$

The associated coefficient matrices $\mathbf{A}_k$ and $\mathbf{g}_k$ in (A.2), $k \geq 3d+1$, are given by

$$\mathbf{A}_k = k\mathbf{I}_{d\times d} - \mathbf{M}, \quad \mathbf{M} = \begin{pmatrix} d & 1 & 0 & \cdots & 0 \\ 0 & d+1 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ & & 0 & 2d-2 & 1 \\ (-1)^{d-1}\frac{(2d)!}{d!} & 0 & \cdots & 0 & 2d-1 \end{pmatrix},$$

and

$$\mathbf{g}_k = \begin{pmatrix} \sum_{0\le\ell<k} u_\ell^{[2]}(\tau) \\ \vdots \\ \sum_{0\le\ell<k} u_\ell^{[d]}(\tau) \\ (-1)^d \rho \sum_{1\le\ell<k} u_\ell^{[1]}(\tau) u_{k-\ell}^{[1]}(\tau) \end{pmatrix}.$$

Due to the existence of complex-conjugate roots, we can find a $d \times d$ matrix $\mathbf{P}$ with entries $\mathbf{P}_{ij} \in \mathbb{C}$ such that

$$\mathbf{P}\mathbf{A}_k\mathbf{P}^{-1} = \begin{pmatrix} k+1 & 0 & \cdots & \cdots & 0 \\ 0 & k-3d & & & \vdots \\ & & k-r_3 & & \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & r_d \end{pmatrix},$$

for $k \in \mathbb{N}$. Here $r_3, r_4, \ldots, r_d$ denote the non-real zeros of the polynomial in (29). By the same norm and same arguments used for two random BSTs, we derive the inequality

$$\max_{1\le j\le d} \left| u_k^{[j]}(\tau) \right| \le \|\boldsymbol{\phi}_k\| \le \|\mathbf{P}\| \|\mathbf{P}^{-1}\| \int_0^\infty e^{-(k-3d)x} \|\mathbf{g}_k(\tau - x)\| \, \mathrm{d}x.$$

Again, by the same arguments used to prove (19), we obtain (A.3) with $s = d$ and $c(s) = 3d$, where the constant B is determined using the techniques of Section 2.3.

The resonance polynomial equals $\det(r\mathbf{I}_{d\times d} - \mathbf{M})$. Direct calculations give the determinant

$$\det\left(r\mathbf{I}_{d\times d} - \mathbf{M}\right) = \frac{(2d-1-r)!}{(d-1-r)!} - \frac{(2d)!}{d!},$$

which is nothing but the resonance polynomial (29).

The reason that the two polynomials are equal is as follows. The distinction between Laurent expansion and the psi-series expansion depends crucially either on the existence of positive integer resonance or on whether a relation such as (9) holds for all k. This is equivalent to asking whether the linear system $\mathbf{A}_k\boldsymbol{\phi}_k = \mathbf{g}_k$ is solvable or not for all k. If the system (A.2) $\mathbf{A}_k\boldsymbol{\phi}_k = \mathbf{g}_k$ is solvable under the condition $\det \mathbf{A}_k \neq 0$ for all k, then by the uniqueness of the solution of (A.2), the solution vectors $\boldsymbol{\phi}_k$'s are constant vectors (independent of τ) and in turn, the series solution $U_1(Z) = \sum_{k\ge0} u_k^{[1]}(\tau) Z^{-d+k-j+1}$ is eventually a Laurent series. On the other hand, if $\det \mathbf{A}_{k_0} \neq 0$ fails to hold for some k_0, then we have the following two cases.

- — The linear system $\mathbf{A}_{k_0}\boldsymbol{\phi} = \mathbf{g}_{k_0}$ has a solution depending on the $d - \operatorname{rank}(\mathbf{A}_{k_0})$ free parameters, and all the remaining constant coefficient vectors $\boldsymbol{\phi}_k$ depend on at least these parameters.
- — The linear system is inconsistent. Hence it can no longer provide a solution to (A.2). The true solution should be solved from (A.2) instead and then all the vector functions $\boldsymbol{\phi}_k(\tau)$, $k \ge k_0$, depend on τ, and the resulting solution $U_1(Z) = Z^{-d} \sum_{k\ge0} u_k^{[1]}(\tau) Z^{k-j+1}$ is indeed a psi-series.

In particular, we see that the characteristic polynomial $\det(r\mathbf{I}_{d\times d} - \mathbf{M})$ equals the polynomial (29) that determines all the possible resonances.

Equality of two random m-ary search trees. The transformed first-order differential system in terms of Z for (31) now has the form

$$\begin{cases} U_j'(Z) = U_{j+1}(Z), & 1 \le j \le m-2, \\ U_{m-1}'(Z) = (1-Z)^{-(m-1)} U_m(Z), & \\ U_{m-1+j}'(Z) = U_{m+j}(Z), & 1 \le j \le m-2, \\ U_{2m-2}'(Z) = (m-1)!^2 \rho^{m-1} U_1(Z)^m. & \end{cases}$$

Then the corresponding infinite system (A.2) has the coefficient matrix $\mathbf{A}_k = k\mathbf{I}_{2(m-1)\times 2(m-1)} - \mathbf{M}$, where

$$\mathbf{M} = \begin{pmatrix} 2 & 1 & 0 & \cdots & & \cdots & & 0 \\ 0 & 3 & 1 & 0 & & & & \vdots \\ \vdots & & \ddots & \ddots & & & & \vdots \\ 0 & & \cdots & & 0 & 2m-2 & & 1 \\ m(2m-1)! & 0 & \cdots & & \cdots & 0 & & 2m-1 \end{pmatrix},$$

and the vector-valued function $\mathbf{g}_k$ is defined by $\mathbf{g}_k = (g_{k,1}, \dots, g_{k,2m-2})^{\mathrm{T}}$ (the superscript T denoting the transpose) with $g_{k,j} = 0$ for $j \ne m-1, 2m-2$, $g_{k,m-1} = \sum_{0 \le j < k} \binom{m-2+k-j}{k-j} u_j^{[m]}$, and $g_{k,2m-2} = \rho^{m-1}(m-1)!^2 \sum_{\substack{i_1+i_2+\cdots+i_m=k \\ 0 \le i_j < k}} u_{i_1}^{[1]} u_{i_2}^{[1]} \cdots u_{i_m}^{[1]}$.

Then similar arguments as those used for (19) lead to the upper bound (A.3) with $s = 2(m-1)$ and $c(s) = 2(m+1)$.

Equality of two random median-of-$(2t+1)$ BSTs. The linear differential system of $4t+2$ equations of (33) is

$$\begin{cases} U_j'(Z) = U_{j+1}(Z), & 1 \le j \le 2t, \\ U_{2t+1}'(Z) = (1-Z)^{-(2t+1)} U_{2t+2}(Z), & \\ U_j'(Z) = U_{j+1}(Z), & 2t+2 \le j \le 4t+1, \\ U_{4t+2}'(Z) = \frac{(2t+1)!^2}{t!^4} \rho \sum_{0 \le i_1, i_2 \le t} \mu(i_1,i_2)(1-Z)^{2t-i_1-i_2} U_{2t+1-i_1}(Z) U_{2t+1-i_2}(Z), & \end{cases}$$

where $\mu(i_1,i_2) := (-1)^{i_1+i_2} t!^4/(i_1! i_2! (t-i_1)!^2 (t-i_2)!^2)$. Let $U_j(Z) = \sum_{k\ge 0} u_k^{[j]}(\tau) Z^{k-j-1}$ for $1 \le j \le 4t+2$, where $u_0^{[j]}(\tau) = (-1)^{j-1} j!(4t+3)!\, t!^4/(\rho\,(2t+1)!^4)$ for $1 \le j \le 2t+1$. Then the coefficient matrix $\mathbf{A}_k = k\mathbf{I}_{(4t+2)\times(4t+2)} - \mathbf{M}$ in (A.2), $k \ge 4t+2$, is given by

$$\mathbf{M} = \begin{pmatrix} 2 & 1 & 0 & \cdots & & & & & 0 \\ 0 & 3 & 1 & \cdots & & & & & 0 \\ \vdots & \ddots & \ddots & \ddots & & & & & \vdots \\ & & & & & & 0 & 4t+2 & 1 \\ 0 & 0 & \cdots & 0 & \frac{2(4t+3)!}{(2t+1)!} & \cdots & & 0 & 4t+3 \end{pmatrix},$$

and the vector-valued function $\mathbf{g}_k = (g_{k,1}, \ldots, g_{k,4t+2})^{\mathrm{T}}$ by $g_{k,j} = 0$ if $j \neq 2t+1, 4t+2$, $g_{k,2t+1} := \sum_{0 \le j < k} \binom{2t+k-j}{k-j} u_j^{[2t+2]}$, and

$$g_{k,4t+2} := \frac{(2t+1)!^2}{t!^4} \rho\mu(0,0) \sum_{\substack{0 \le \ell \le k-j \\ 1 \le j \le \min\{k,2t\}}} (-1)^j \binom{2t}{j} u_k^{[2t+1]} u_{k-j-\ell}^{[2t+1]}$$
$$+ \sum_{\substack{1 \le s \le \min\{k,2t\} \\ 0 \le i \le \min\{s,t\}}} \mu(i, s-i) \sum_{\substack{0 \le \ell \le k-j \\ 1 \le j \le \min\{k-s,2t-s\}}} (-1)^j \binom{2t-s}{j} u_k^{[2t+1-i]} u_{k-s-j-\ell}^{[2t+1+i-s]}.$$

The same method of proof used for (19) yields the upper bound (A.3) with $s = 4t+2$ and $c(s) = 2$ when $t \ge 1$.

B Proof of the incompatibility of the resonance $r = 2$ for random median-of-$(2t+1)$ BSTs

Since the resonance $r = 2$ does not depend on t, the incompatibility of the resonance $r = 2$ can be directly checked, which we now do. Let $U(Z) := F(z)$, where F satisfies the ODE (33) and $Z = 1 - z/\rho$. Then the ODE (33) can be rewritten as

$$\left((1-Z)^{2t+1} U^{(2t+1)}(Z)\right)^{(2t+1)} = C_{t,\rho} \left(\left((1-Z)^t U^{(t)}(Z)\right)^{(t)}\right)^2, \tag{A.4}$$

where all derivatives are with respect to Z and $C_{t,\rho} := (2t+1)!^2 \rho / t!^4$.

Note that for any $s \in \mathbb{N}$ and a formal Laurent expansion $f(Z) = \sum_{k \ge 0} u_k Z^{k-\alpha}$

$$\left((1-Z)^s f^{(s)}(Z)\right)^{(s)} = \sum_{k \ge 0} (k-\alpha-s)^{\underline{s}} Z^{k-2s-\alpha} \sum_{0 \le j \le s} (-1)^j \binom{s}{j} (k-\alpha-j)^{\underline{s}} u_{k-j},$$

where $u_j := 0, j < 0$ and $x^{\underline{s}} := x(x-1)\cdots(x-s+1)$. Substituting this into (A.4) and $s = 2t+1$, we have

$$\sum_{k \ge 0} (k-\alpha-(2t+1))^{\underline{2t+1}} Z^{k-4t-2\alpha} \sum_{0 \le j \le 2t+1} (-1)^j \binom{2t+1}{j} (k-\alpha-j)^{\underline{2t+1}} u_{k-j}$$
$$= C_{t,\rho} \sum_{k \ge 0} Z^{k-4t-2-\alpha} \sum_{0 \le \ell \le k} \chi_k \chi_{k-\ell},$$

where $\chi_k := (k-\alpha-t)^{\underline{t}} \sum_{0 \le j \le t} (-1)^j \binom{t}{j} (k-\alpha-j)^{\underline{t}} u_{k-j}$. Equating the dominant term (with $k = 0$) leads to the obvious solution $\alpha = 2$. Consider now the relation

$$(k-\alpha-(2t+1))^{\underline{2t+1}} \sum_{0 \le j \le 2t+1} (-1)^j \binom{2t+1}{j} (k-\alpha-j)^{\underline{2t+1}} u_{k-j} = C_{t,\rho} \sum_{0 \le \ell \le k} \chi_k \chi_{k-\ell}.$$

From this, we get $\rho u_0 = (4t+3)!t!^4/(2t+1)!^4$, $u_1 = -2(t+1)^2 u_0/(6t+5)$, and

$$\begin{aligned}0\cdot u_2 &= C_{t,\rho}(2t)!^2\left(\left((2t+1)(t+1)\binom{t}{2}+t^2(t+1)^2\right)u_0^2+u_1^2-t(4t+3)u_0u_1\right)\\ &\quad+(4t+1)!(2t+1)^2u_1-(4t+1)!(2t+1)(2t+2)\binom{2t+1}{2}u_0\\ &= -\frac{(4t+2)!(t+1)}{4(6t+5)^2}u_0\left(216t^4+522t^3+437t^2+141t+12\right)\neq 0,\end{aligned}$$

since $t \geq 1$. This proves the incompatibility of the resonance $r=2$ for all $t \geq 1$.

C Asymptotics of the moment generating function of partial match in random quadtrees

We prove (36), starting from Hankel's integral representation of the Gamma function

$$\frac{1}{\Gamma(w)} = \frac{1}{2\pi \mathrm{i}}\int_{\mathscr{H}_0} e^s s^{-w}\,\mathrm{d}s \qquad (w\in\mathbb{C}),$$

where $\mathscr{H}_0$ starts at $-\infty$, encircles the origin once counter-clockwise and returns to its starting point. For definiteness, we may take

$$\mathscr{H}_0 = \{s = xe^{\pm\pi \mathrm{i}} \,:\, R_0 \leq x < \infty\}\cup\{s = R_0e^{\theta \mathrm{i}} \,:\, -\pi\leq\theta\leq\pi\} \qquad (R_0>0).$$

This gives

$$M(z) := \mathbb{E}(e^{Xz}) = \frac{1}{2\pi\mathrm{i}}\int_{\mathscr{H}_0} e^s s^{-1}X(z/s^v)\,\mathrm{d}s,$$

where $X(z)$ satisfies the ODE (34). Note that M is an entire function of order $1/v > 1$ and of type $\rho^{-1/v}$.

Let $z = |z|e^{\varphi\mathrm{i}}$, $|z|>0$ and $|\varphi| < v\pi/2$, where $v = (\sqrt{17}-3)/2$. The condition on $\arg z$ implies that the dominant singularity $s = (z/\rho)^{1/v}$ of the integrand lies in the half-plane $\Re(s) > 0$ (in which $e^s \to \infty$ with z). On the other hand, if $|\arg(-z)| < \pi - v\pi/2$, then one expects that $M(z) \to 0$ with z, but the exact determination of the rate is more delicate. The situation here is similar to the Mittag-Leffler function $\sum_{j\geq 0} z^j/\Gamma(aj+1)$; see [17, Ch. 18.1].

The change of variables $z/s^v \mapsto s$ gives

$$M(z) = \frac{1}{2\pi \mathrm{i} v}\int_{\mathscr{H}_1} e^{z^{1/v}s^{-1/v}} s^{-1}X(s)\,\mathrm{d}s,$$

where $\mathscr{H}_1$ is the path described by

$$\mathscr{H}_1 = \{s = xe^{(\varphi\pm v\pi)\mathrm{i}} \,:\, 0\leq x\leq R_1\}\cup\{s = R_1e^{(\varphi+v\theta)\mathrm{i}} \,:\, -\pi\leq\theta\leq\pi\}.$$

Here $0 < R_1 < \rho$. We then approach in a way similar to the singularity analysis (see [21]) by deforming the contour $\mathscr{H}_1$ into $\mathscr{H}_2$, where $\mathscr{H}_2$ is of the same shape as $\mathscr{H}_1$ but with larger radius for the circular part $|s| = R_2 = \rho + \varepsilon$ and avoiding the cut from $s = \rho$ to ∞ (in the style of [21]). Symbolically,

$$\begin{aligned}\mathscr{H}_2 = \Gamma_\rho &\cup \{s = xe^{(\varphi \pm v\pi)\mathrm{i}} \,:\, 0 \le x \le R_2\} \\ &\cup \{s = R_2 e^{(\varphi + v\theta)\mathrm{i}} \,:\, -\pi \le \theta \le \pi \text{ and } |\theta - \varphi/v| \ge c_z\},\end{aligned}$$

where $c_z := |z|^{-1/v}$ and Γ_ρ is any contour joining the two points $R_2 e^{-c_z \mathrm{i}}$ and $R_2 e^{c_z \mathrm{i}}$ and lying inside the cut region described by other parts of $\mathscr{H}_2$.

The remaining analysis is then easy because the main contribution to $M(z)$ comes from Γ_ρ on which we can apply the local expansion (35) of $X(z)$, the other parts being negligible

$$M(z) = \frac{1}{2\pi \mathrm{i} v} \int_{\Gamma_\rho} e^{z^{1/v} s^{-1/v}} s^{-1} X(s) \, \mathrm{d}s + O\left(e^{\Re(z/(\rho+\varepsilon))^{1/v}}\right).$$

By making first the change of variables $\rho(1 - s) \mapsto s$, using the expansion (35), and then another change of variables $(z/\rho)^{1/v} s/v \mapsto s$, we deduce that

$$\begin{aligned}M(z) = \frac{e^{(z/\rho)^{1/v}}}{2\pi \mathrm{i}} \int_{\Gamma_0} e^s \Bigg(3\left(\frac{z}{\rho}\right)^{1/v} s^{-2} + \frac{9}{5} s^{-1} + \sum_{2 \le j \le 7} \left(\bar{c}_j(s) + \tilde{c}_j(s) \log \frac{z}{\rho}\right) \left(\frac{z}{\rho}\right)^{-j/v} \\ + \frac{936}{21875} \left(\frac{z}{\rho}\right)^{-5/v} s^4 \log s + O\left(|z|^{-6/v} |s|^5 |\log s|\right) \Bigg) \mathrm{d}s,\end{aligned}$$

where Γ_0 denotes the transformed contour of Γ_ρ and the c_j''s are polynomials of s whose exact values matter less. Extending the contour to infinity and then evaluating the individual terms by Hankel's integral representation of the Gamma function, we obtain

$$M(z) = e^{(z/\rho)^{1/v}} \left(3\left(\frac{z}{\rho}\right)^{1/v} + \frac{9}{5} - \frac{22464}{21875} \left(\frac{z}{\rho}\right)^{-5/v} + O\left(|z|^{-6/v}\right) \right),$$

where we also used the formula

$$\frac{1}{2\pi \mathrm{i}} \int_{\mathscr{H}_0} e^s s^4 \log s \, \mathrm{d}s = -\frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{\Gamma(x)} \bigg|_{x=-4} = -24.$$

This completes the proof of (36).