The connectivity-profile of random increasing k-trees

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Abstract

Random increasing k-trees represent an interesting, useful class of strongly dependent graphs for which analytic-combinatorial tools can be successfully applied. We study in this paper a notion called connectivity-profile and derive asymptotic estimates for it; some interesting consequences will also be given.

1 Introduction

A k-tree is a graph reducible to a k-clique by successive removals of a vertex of degree k whose neighbors form a k-clique. This class of k-trees has been widely studied in combinatorics (for enumeration and characteristic properties [5, 29]), in graph algorithms (many NP-complete problems on graphs can be solved in polynomial time on k-trees [2]), and in many other fields where k-trees were naturally encountered (see [2]). By construction, vertices in such structures are remarkably close, reflecting a highly strong dependent graph structure, and they exhibit with no surprise the scale-free property [20], yet somewhat unexpectedly many properties of random k-trees can be dealt with by standard combinatorial, asymptotic and probabilistic tools, thus providing an important model of synergistic balance between mathematical tractability and the predictive power for practical-world complex networks.

While the term "k-trees" is not very informative and may indeed be misleading to some extent, they stand out by their underlying tree structure, related to their recursive definition, which facilitates the analysis of the properties and the exploration of the structure. Indeed, for $k = 1$, k-trees are just trees, and for $k \geq 2$ a bijection [11] can be explicitly defined between k-trees and a non trivial simple family of trees.

The process of generating a k-tree begins with a k-clique, which is itself a k-tree; then the k-tree grows by linking a new vertex to every vertex of an existing k-clique, and to these vertices only. The same process continues; see Figure 1 for an illustration. Such a simple process is reminiscent of several other models proposed in the literature such as k-DAGs [13], random circuits [3], preferential attachment [4, 7, 21], and many other models (see, for example, [6, 17, 25]). While the construction rule

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in each of these models is very similar, namely, linking a new vertex to k existing ones, the mechanism of choosing the existing k vertices differs from one case to another, resulting in very different topology and dynamics.

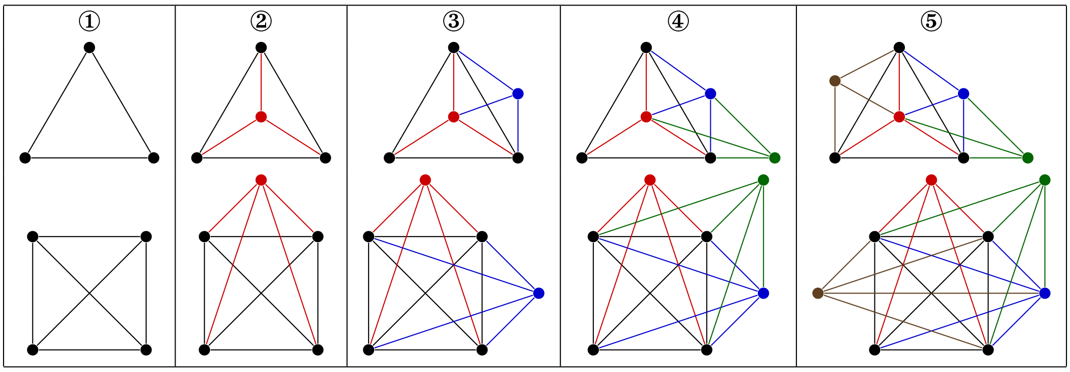


Figure 1: *The first few steps of generating a* 3*-tree and a* 4*-tree. Obviously, these graphs show the high connectivity of k-trees.*

Restricting to the procedure of choosing a k-clique each time a new vertex is added, there are several variants of k-trees proposed in the literature depending on the modeling needs. So k-trees can be either labeled [5], unlabeled [22], increasing [32], planar [32], non-planar [5], or plane [26], etc.

For example, the family of random Apollonian networks, corresponding to planar 3-trees, has recently been employed as a model for complex networks [1, 32]. In these frameworks, since the exact topology of the real networks is difficult or even impossible to describe, one is often led to the study of models that present similarities to some observed properties such as the degree of a node and the distance between two nodes of the real structures.

For the purpose of this paper, we distinguish between two models of random labeled non-plane k-trees; by non-plane we mean that we consider these graphs as given by a set of edges (and not by its graphical representation):

– *random simply-generated k-trees*, which correspond to a uniform probability distribution on this class of k-trees, and

– *random increasing k-trees*, where we consider the iterative generation process: at each time step, all existing k-cliques are equally likely to be selected and the new vertex is added with a label which is greater than the existing ones.

The two models are in good analogy to the simply-generated family of trees of Meir and Moon [24] marked specially by the functional equation $f(z) = z\Phi(f(z))$ for the underlying enumerating generating function, and the increasing family of trees of Bergeron et al. [10], characterized by the differential equation $f'(z) = \Phi(f(z))$. Very different stochastic behaviors have been observed for these families of trees. While similar in structure to these trees, the analytic problems on random k-trees we are dealing with here are however more involved because instead of a scalar equation (either functional, algebraic, or differential), we now have a system of equations.

It is known that random trees in the family of increasing trees are often less skewed, less slanted in shape, a typical description being the logarithmic order for the distance of two randomly chosen nodes; this is in sharp contrast to the square-root order for random trees belonging to the simply-generated family; see for example [10, 14, 19, 23, 24]. Such a contrast has inspired and stimulated much recent research. Indeed, the majority of random trees in the literature of discrete probability, analysis of algorithms, and random combinatorial structures are either $\log n$-trees or $\sqrt{n}$-trees, n being the tree

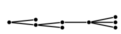
Model / Properties	Simply-generated structures	Increasing structures
Combinatorial description	$\mathcal{T}_s = \text{Set}(\mathcal{Z} \times \mathcal{T}_s^k)$	$\mathcal{T} = \text{Set}(\mathcal{Z}^{\square} \times \mathcal{T}^k)$
Generating function	$T_s(z) = \exp(zT_s^k(z))$	$T'(z) = T^k(z)$
Expansion near singularity	$T_s(z) = \tau - h\sqrt{1 - z/\rho} + \dots$	$T(z) = (1 - kz)^{-1/k}$
Mean distance of nodes	$O(\sqrt{n})$	$O(\log n)$
Degree distribution	Power law with exp. tails	Power law [20]
Root-degree distribution	Power law with exp. tails	Stable law (Theorem 7)
Expected Profile	Rayleigh limit law	Gaussian limit law (8)

Table 1: *The contrast of some properties between random simply-generated k-trees and random increasing k-trees. Here $\mathcal{Z}$ denotes a node and $\mathcal{Z}^{\square}$ means a marked node.*

size. While the class of $\sqrt{n}$-trees have been extensively investigated by probabilists and combinatorialists, $\log n$-trees are comparatively less addressed, partly because most of them were encountered not in probability or in combinatorics, but in the analysis of algorithms.

Table 1 presents a comparison of the two models: the classes $\mathcal{T}_s$ and $\mathcal{T}$, corresponding respectively to simply-generated k-trees and increasing k-trees. The results concerning simple k-trees are given in [11, 12], and those concerning increasing k-trees are derived in this paper (except for the power law distribution [20]). We start with the specification, described in terms of operators of the symbolic method [18]. A structure of $\mathcal{T}_s$ is a set of k structures of the same type, whose roots are attached to a new node: $\mathcal{T}_s = \text{Set}(\mathcal{Z} \times \mathcal{T}_s^k)$, while a structure of $\mathcal{T}$ is an increasing structure, in the sense that the new nodes get labels that are smaller than those of the underlying structure (this constraint is reflected by the box-operator) $\mathcal{T} = \text{Set}(\mathcal{Z}^{\square} \times \mathcal{T}^k)$. The analytic difference immediately appears in the enumerative generating functions that translate the specifications: the simply-generated structures are defined by $T_s(z) = \exp(zT_s^k(z))$ and corresponding increasing structures satisfy the differential equation $T'(z) = T^k(z)$. These equations lead to a singular expansion of the square-root type in the simply-generated model, and a singularity in $(1 - kz)^{-1/k}$ in the increasing model. Similar analytic differences arise in the bivariate generating functions of shape parameters.

The expected distance between two randomly chosen vertices or the average path length is one of the most important shape measures in modeling complex networks as it indicates roughly how efficient the information can be transmitted through the network. Following the same $\sqrt{n}$-vs-$\log n$ pattern, it is of order $\sqrt{n}$ in the simply-generated model, but $\log n$ in the increasing model. Another equally important parameter is the degree distribution of a random vertex: its limiting distribution is a power law with exponential tails in the simply-generated model of the form $d^{-3/2}\rho_k^d$, in contrast to a power-law in the increasing model of the form $d^{-1-k/(k-1)}$, d denoting the degree [20]. As regards the degree of the root, its asymptotic distribution remains the same as that of any vertex in the simply-generated model, but in the increasing model, the root-degree distribution is different, with an asymptotic stable law (which is Rayleigh in the case $k = 2$); see Theorem 7.

Our main concern in this paper is the connectivity-profile. Recall that the profile of an usual tree is the sequence of numbers, each enumerating the total number of nodes with the same distance to the root. For example, the tree  has the profile $\{1, 2, 2, 1, 3\}$. Profiles represent one of the richest shape measures and they convey much information regarding particularly the silhouette. On random trees, they have been extensively studied recently; see [8, 15, 16, 19, 21, 23, 27]. Since k-trees have many cycles for $k \geq 2$, we call the profile of the transformed tree (see next section) *the connectivity-profile* as it measures to some extent the connectivity of the graph. Indeed this connectivity-profile corresponds to the profile of the "shortest-path tree" of a k-tree, as defined by Proskurowski [28], which is nothing more than the result of a Breadth First Search (BFS) on the graph. Moreover, in the domain of complex networks,

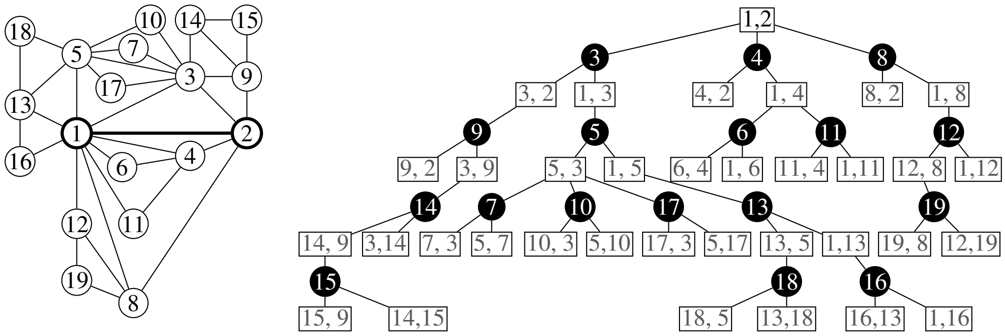


Figure 2: *A* 2*-tree (left) and its corresponding increasing tree representation (right).*

this kind of BFS trees is an important object; for example, it describes the results of the `traceroute` measuring tool [30, 31] in the study of the topology of the Internet.

We will derive precise asymptotic approximations to the expected connectivity-profile of random increasing k-trees, the major tools used being based on the resolution of a system of differential equations of Cauchy-Euler type (see [9]). In particular, the expected number of nodes at distance d from the root follows asymptotically a Gaussian distribution, in contrast to the Rayleigh limit distribution in the case of simply-generated k-trees. Also the limit distribution of the number of nodes with distance d to the root will be derived when d is bounded. Note that when $d = 1$, the number of nodes at distance 1 to the root is nothing but the degree of the root.

This paper is organized as follows. We first present the definition and combinatorial specification of random increasing k-trees in Section 2, together with the enumerative generating functions, on which our analytic tools will be based. We then present two asymptotic approximations to the expected connectivity-profile in Section 3, one for $d = o(\log n)$ and the other for $d \to \infty$ and $d = O(\log n)$. Interesting consequences of our results will also be given. The limit distribution of the connectivity-profile in the range when $d = O(1)$ is then given in Section 4.

2 Random increasing k-trees and generating functions

Since k-trees are graphs full of cycles and cliques, the key step in our analytic-combinatorial approach is to introduce a bijection between k-trees and a suitably defined class of trees (*bona fide* trees!) for which generating functions can be derived. This approach was successfully applied to simply-generated family of k-trees in [11], which leads to a system of algebraic equations. The bijection argument used there can be adapted *mutatis mutandis* here for increasing k-trees, which then yields a system of differential equations through the bijection with a class of increasing trees [10].

Increasing k-trees and the bijection. Recall that a k-clique is a set of k mutually adjacent vertices.

Definition 1 *An increasing k-tree is defined recursively as follows. A k-clique in which each vertex gets a distinct label from $\{1, \dots, k\}$ is an increasing k-tree of k vertices. An increasing k-tree with $n > k$ vertices is constructed from an increasing k-tree with $n - 1$ vertices by adding a vertex labeled n and by connecting it by an edge to each of the k vertices in an existing k-clique.*

By *random increasing k trees*, we assume that all existing k-cliques are equally likely each time a new vertex is being added. One sees immediately that the number T_n of increasing k-trees of $n + k$ nodes is given by $T_n = \prod_{0 \le i < n}(ik + 1)$.

Note that if we allow any permutation on all labels, we obtain the class of simply-generated k-trees where monotonicity of labels along paths fails in general.

Combinatorially, simply-generated k-trees are in bijection [11] with the family of trees specified by $\mathcal{K}_s = \mathcal{Z}^k \times \mathcal{T}_s$, where $\mathcal{T}_s = \text{Set}(\mathcal{Z} \times \mathcal{T}_s^k)$. Given a rooted k-tree G of n vertices, we can transform G into a tree T, with the root node labeled $\{1, \dots, k\}$, by the following procedure. First, associate a white node to each k-clique of G and a black node to each $(k+1)$-clique of G. Then add a link between each black node and all white nodes associated to the k-cliques it contains. Each black node is labeled with the only vertex not appearing in one of the black nodes above it or in the root. The last step in order to complete the bijection is to order the k vertices of the root and propagate this order to the k sons of each black node. This constructs a tree from a k-tree (see Figure 2); conversely, we can obtain the k-tree through a simple traversal of the tree.

Such a bijection translates directly to increasing k-trees by restricting the class of corresponding trees to those respecting a monotonicity constraint on the labels, namely, on any path from the root to a leaf the labels are in increasing order. This yields the combinatorial specification of the class of increasing trees $\mathcal{T} = \text{Set}(\mathcal{Z}^{\Box} \times \mathcal{T}^k)$. An increasing k-tree is just a tree in $\mathcal{T}$ together with the sequence $\{1, \dots, k\}$ corresponding to the labels of the root-clique[1]. A tree in $\mathcal{K}$ is thus completely determined by its $\mathcal{T}$ component, giving $\mathcal{K}_{n+k} \equiv \mathcal{T}_n$. For example figure 2 shows a 2-tree with 19 vertices and its tree representation with 17 black nodes. In the rest of this paper we will thus focus on class $\mathcal{T}$.

Generating functions. Following the bijection, we see that the complicated dependence structure of k-trees is now completely described by the class of increasing trees specified by $\mathcal{T} = \text{Set}(\mathcal{Z}^{\Box} \times \mathcal{T}^k)$. For example, let $T(z) := \sum_{n \geq 0} T_n z^n / n!$ denote the exponential generating function of the number T_n of increasing k-trees of $n + k$ vertices. Then the specification translates into the equation

$$T(z) = \exp\left(\int_0^z T^k(x)\,\mathrm{d}x\right),$$

or, equivalently, $T'(z) = T^{k+1}(z)$ with $T(0) = 1$, which is solved to be

$$T(z) = (1 - kz)^{-1/k},$$

we then check that $T_n = \prod_{0 \leq i < n}(ik + 1)$.

If we mark the number of neighbors of the root-node in $\mathcal{T}$ by u, we obtain

$$T(z, u) = \exp\left(u \int_0^z T(x) T^{k-1}(x, u)\,\mathrm{d}x\right),$$

where the coefficients $n![u^\ell z^n]T(z, u)$ denote the number of increasing k-trees of size $n + k$ with root degree equal to $k + \ell - 1$. Taking derivative with respect to z on both sides and then solving the equation, we get the closed-form expression

$$T(z, u) = \left(1 - u(1 - (1 - kz)^{1-1/k})\right)^{-1/(k-1)}. \tag{1}$$

Since k-trees can be transformed into ordinary increasing trees, the profiles of the transformed trees can be naturally defined, although they do not correspond to simple parameters on k-trees. While the study of profiles may then seem artificial, the results do provide more insight on the structure of random k-trees. Roughly, we expect that all vertices on k-trees are close, one at most of logarithmic order away from the other. The fine results we derive provide in particular an upper bound for that.

[1] We call *root-clique* the clique composed by the k vertices $(1, \dots, k)$. The increasing nature of the k-trees guarantees that these vertices always form a clique. We call *root-vertex* the vertex with label 1.

Let $X_{n;d,j}$ denote the number of nodes at distance d from j vertices of the root-clique in a random k-tree of $n+k$ vertices. Let $T_{d,j}(z,u) = \sum_{n\geq 0} T_n \mathbb{E}(u^{X_{n;d,j}}) z^n/n!$ denote the corresponding bivariate generating function.

Theorem 1 *The generating functions $T_{d,j}$'s satisfy the differential equations*

$$\frac{\partial}{\partial z} T_{d,j}(z,u) = u^{\delta_{d,1}} T_{d,j-1}^{j}(z,u) T_{d,j}^{k-j+1}(z,u), \tag{2}$$

with the initial conditions $T_{d,j}(0,u) = 1$ for $1 \leq j \leq k$, where $\delta_{a,b}$ denotes the Kronecker function, $T_{0,k}(z,u) = T(z)$ and $T_{d,0}(z,u) = T_{d-1,k}(z,u)$.

Proof. The theorem follows from

$$T_{d,j}(z,u) = \exp\left(u^{\delta_{d,1}} \int_0^z T_{d,j-1}^{j}(x,u) T_{d,j}^{k-j}(x,u)\,\mathrm{d}x\right),$$

with $T_{d,j}(z,1) = T(z)$. □

For operational convenience, we normalize all z by z/k and write $\tilde{T}(z) := T(z/k) = (1-z)^{-1/k}$. Similarly, we define $\tilde{T}_{d,j}(z,u) := T_{d,j}(z/k,u)$ and have, by (2),

$$\frac{\partial}{\partial z} \tilde{T}_{d,j}(z,u) = \frac{u^{\delta_{d,1}}}{k} \tilde{T}_{d,j-1}^{j}(z,u) \tilde{T}_{d,j}^{k-j+1}(z,u), \tag{3}$$

with $\tilde{T}_{d,j}(1,z) = \tilde{T}(z)$, $\tilde{T}_{0,k}(z,u) = \tilde{T}(z)$ and $\tilde{T}_{d,0}(z,u) = \tilde{T}_{d-1,k}(z,u)$.

3 Expected connectivity-profile

We consider the expected connectivity-profile $\mathbb{E}(X_{n;d,j})$ in this section. Observe first that

$$\mathbb{E}(X_{n;d,j}) = \frac{k^n [z^n] \tilde{M}_{d,j}(z)}{T_n},$$

where $\tilde{M}_{d,j}(z) := \partial \tilde{T}_{d,j}(z,u)/(\partial u)|_{u=1}$. It follows from (3) that

$$\tilde{M}_{d,j}'(z) = \frac{1}{k(1-z)} \left((k-j+1)\tilde{M}_{d,j}(z) + j\tilde{M}_{d,j-1}(z) + \delta_{d,1}\tilde{T}(z)\right). \tag{4}$$

This is a standard differential equation of Cauchy-Euler type whose solution is given by (see [9])

$$\tilde{M}_{d,j}(z) = \frac{(1-z)^{-(k-j+1)/k}}{k} \int_0^z (1-x)^{-(j-1)/k} \left(j\tilde{M}_{d,j-1}(x) + \delta_{d,1}\tilde{T}(x)\right) \mathrm{d}x,$$

since $\tilde{M}_{d,j}(0) = 0$. Then, starting from $\tilde{M}_{0,k} = 0$, we get

$$\tilde{M}_{1,1}(z) = \frac{1}{k-1}\left(\frac{1}{1-z} - \frac{1}{(1-z)^{1/k}}\right) = \frac{\tilde{T}^k(z) - \tilde{T}(z)}{k-1}.$$

Then by induction, we get

$$\tilde{M}_{d,j}(z) \sim \frac{j}{(k-1)(d-1)!} \cdot \frac{1}{1-z} \log^{d-1} \frac{1}{1-z} \qquad (1 \leq j \leq k; d \geq 1; z \sim 1).$$

So we expect, by singularity analysis, that

$$\mathbb{E}(X_{n;d,j}) \sim \Gamma(1/k)\frac{j}{k-1} \cdot \frac{(\log n)^{d-1}}{(d-1)!}\, n^{1-1/k},$$

for large n and fixed d, k and $1 \leq j \leq k$. We can indeed prove that the same asymptotic estimate holds in a larger range.

Theorem 2 *The expected connectivity-profile* $\mathbb{E}(X_{n;d,j})$ *satisfies for* $1 \leq d = o(\log n)$

$$\mathbb{E}(X_{n;d,j}) \sim \Gamma(1/k)\frac{j}{k-1} \cdot \frac{(\log n)^{d-1}}{(d-1)!}\, n^{1-1/k}, \tag{5}$$

uniformly in d*, and for* $d \to \infty$*,* $d = O(\log n)$*,*

$$\mathbb{E}(X_{n;d,j}) \sim \frac{\Gamma(1/k) h_{j,1}(\rho)\rho^{-d} n^{\lambda_1(\rho)-1/k}}{\Gamma(\lambda_1(\rho))\sqrt{2\pi(\rho\lambda_1'(\rho) + \rho^2\lambda_1''(\rho))\log n}} \tag{6}$$

where $\rho = \rho_{n,d} > 0$ *solves the equation* $\rho\lambda_1'(\rho) = d/\log n$*,* $\lambda_1(w)$ *being the largest zero (in real part) of the equation* $\prod_{1\leq\ell\leq k}(\theta - \ell/k) - k!w/k^k = 0$ *and satisfies* $\lambda_1(1) = (k+1)/k$*.*

An explicit expression for the $h_{j,1}$'s is given as follows. Let $\lambda_1(w), \ldots, \lambda_k(w)$ denote the zeros of the equation $\prod_{1\leq\ell\leq k}(\theta - \ell/k) - k!w/k^k = 0$. Then for $1 \leq j \leq k$

$$h_{j,1}(w) = \frac{j!w(w-1)}{(k\lambda_1(w)-1)\left(\sum_{1\leq s\leq k}\frac{1}{k\lambda_1(w)-s}\right)\prod_{k-j+1\leq s\leq k+1}(k\lambda_1(w)-s)}. \tag{7}$$

The theorem cannot be proved by the above inductive argument and our method of proof consists of the following steps. First, the bivariate generating functions $\mathscr{M}_j(z,w) := \sum_{d\geq 1}\tilde{M}_{d,j}(z)w^d$ satisfy the linear system

$$\left((1-z)\frac{\mathrm{d}}{\mathrm{d}z} - \frac{k-j+1}{k}\right)\mathscr{M}_j = \frac{j}{k}\mathscr{M}_{j-1} + \frac{w\tilde{T}}{k} \qquad (1 \leq j \leq k).$$

Second, this system is solved and has the solutions

$$\mathscr{M}_j(z,w) = \sum_{1\leq j\leq k} h_{j,m}(w)(1-z)^{-\lambda_m(w)} - \frac{w-(w-1)\delta_{k,j}}{k}\,\tilde{T}(z),$$

where the $h_{j,m}$ have the same expression as $h_{j,1}$ but with all $\lambda_1(w)$ in (7) replaced by $\lambda_m(w)$. While the form of the solution is well anticipated, the hard part is the calculations of the coefficient-functions $h_{j,m}$. Third, by singularity analysis and a delicate study of the zeros, we then conclude, by saddle-point method, the estimates given in the theorem.

Corollary 3 *The expected degree of the root* $\mathbb{E}(X_{n,1,j})$ *satisfies*

$$\mathbb{E}(X_{n,1,j}) \sim \Gamma(1/k)\frac{j}{k-1}\, n^{1-1/k} \qquad (1 \leq j \leq k).$$

This estimate also follows easily from (1).

Let $H_k := \sum_{1\leq\ell\leq k} 1/\ell$ denote the harmonic numbers and $H_k^{(2)} := \sum_{1\leq\ell\leq k} 1/\ell^2$.

Corollary 4 *The expected number of nodes at distance $d = \left\lfloor \frac{1}{kH_k}\log n + x\sigma\sqrt{\log n}\right\rfloor$ from the root, where $\sigma = \sqrt{H_k^{(2)}/(kH_k^3)}$, satisfies, uniformly for $x = o((\log n)^{1/6})$,*

$$\mathbb{E}(X_{n;d,j}) \sim \frac{ne^{-x^2/2}}{\sqrt{2\pi\sigma^2\log n}}. \tag{8}$$

This Gaussian approximation justifies the last item corresponding to increasing trees in Table 1.

Note that $\lambda_1(1) = (k+1)/k$ and $\alpha = d/\log n \sim 1/(kH_k)$. In this case, $\rho = 1$ and

$$\rho\lambda_1'(\rho) = \frac{1}{\sum_{1\le\ell\le k}\frac{1}{\lambda_1(\rho)-\frac{\ell}{k}}},$$

which implies that $\lambda_1(\rho) - 1/k - \alpha\log\rho \sim 1$.

Corollary 5 *Let $\mathscr{H}_{n;d,j} := \max_d X_{n;d,j}$ denote the height of a random increasing k-tree of $n+k$ vertices. Then*

$$\mathbb{E}(\mathscr{H}_n) \le \alpha_+\log n - \frac{\alpha_+}{2(\lambda_1(\alpha_+)-\frac{1}{k})}\log\log n + O(1),$$

where $\alpha_+ > 0$ is the solution of the system of equations

$$\begin{cases} \dfrac{1}{\alpha_+} = \displaystyle\sum_{1\le\ell\le k}\frac{1}{v-\frac{\ell}{k}}, \\ v - \dfrac{1}{k} - \alpha_+\displaystyle\sum_{1\le\ell\le k}\log\left(\frac{k}{\ell}v - 1\right) = 0. \end{cases}$$

Table 2 gives the numerical values of α_+ for small values of k. For large k, one can show that $\alpha_+ \sim$

k	2	3	4	5	6
α_+	1.085480	0.656285	0.465190	0.358501	0.290847
k	7	8	9	10	20
α_+	0.244288	0.210365	0.184587	0.164356	0.077875

Table 2: *Approximate numerical values of α_+.*

$1/(k\log 2)$ and $\lambda_1(\alpha_+) \sim 2$.

Corollary 5 justifies that the mean distance of random k-trees are of logarithmic order in size, as stated in Table 1.

Corollary 6 *The width $\mathscr{W}_{n;d,j} := \max_d X_{n;d,j}$ is bounded below by*

$$\mathbb{E}(\mathscr{W}_n) = \mathbb{E}(\max_d X_{n,d}) \ge \max_d \mathbb{E}(X_{n,d}) \asymp \frac{n}{\sqrt{\log n}}.$$

We may conclude briefly from all these results that *in the transformed increasing trees of random increasing k-trees, almost all nodes are located in the levels with $d = \frac{1}{kH_k}\log n + O(\sqrt{\log n})$, each with $n/\sqrt{\log n}$ nodes.*

4 Limiting distributions

With the availability of the bivariate generating functions (2), we can proceed further and derive the limit distribution of $X_{n;d,j}$ in the range where $d = O(1)$. The case when $d \to \infty$ is much more involved; we content ourselves in this extended abstract with the statement of the result for bounded d.

Theorem 7 *The random variables $X_{n;d,j}$, when normalized by their mean orders, converge in distribution to*

$$\frac{X_{n;d,j}}{n^{1-1/k}(\log n)^{d-1}/(d-1)!} \xrightarrow{d} \Xi_{d,j}, \tag{9}$$

where

$$\begin{aligned}\mathbb{E}(e^{\Xi_{d,j}u}) &= \Gamma(\tfrac{1}{k}) \sum_{m\geq 0} \frac{c_{d,j,m}}{m!\Gamma(m(1-1/k)+1/k)} u^m \\ &= \frac{\Gamma(\frac{1}{k})}{2\pi i} \int_{-\infty}^{(0+)} e^{\tau}\tau^{-1/k} C_{d,j}\left(\tau^{-1+1/k}u\right) d\tau,\end{aligned}$$

and $C_{d,j}(u) := 1 + \sum_{m\geq 1} c_{d,j,m}u^m/m!$ satisfies the system of differential equations

$$(k-1)uC'_{d,j}(u) + C_{d,j}(u) = C_{d,j}(u)^{k+1-j}C_{d,j-1}(u)^j \qquad (1 \leq j \leq k), \tag{10}$$

with $C_{d,0} = C_{d-1,k}$. Here the symbol $\int_{-\infty}^{(0+)}$ denotes any Hankel contour starting from $-\infty$ on the real axis, encircling the origin once counter-clockwise, and returning to $-\infty$.

We indeed prove the convergence of all moments, which is stronger than weak convergence; also the limit law is uniquely determined by its moment sequence.

So far only in special cases do we have explicit solution for $C_{1,j}$: $C_{1,1}(u) = (1+u)^{-1/(k-1)}$ and

$$C_{1,2}(u) = \begin{cases} \frac{e^{1/(1+u)}}{1+u}, & \text{if } k = 2; \\ \frac{1}{1+u^{1/2}\arctan(u^{1/2})}, & \text{if } k = 3. \end{cases}$$

Note that the result (9) when $d = 0$ can also be derived directly by the explicit expression (1). In particular, when $k = 2$, the limit law is Rayleigh.

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